Type Theory and Univalent Foundations

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Equality in Type Theory

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash 1_a : \mathsf{Id}_A\ a\ a}$$

$$\frac{\Gamma \vdash d : C(a, \mathsf{Ref}\ a)}{\Gamma \vdash \mathsf{J}'\ d : (\Pi x : A)(\Pi \alpha : \mathsf{Id}_A\ a\ x) C(x, \alpha)}$$

$$\mathsf{J}'\ d\ a\ (\mathsf{Ref}\ a) = d : C(a, \mathsf{Ref}\ a)$$

Equality in Type Theory

Write $a =_A x$ for $\mathsf{Id}_A\ a\ x$

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash 1_a : a =_A a}$$

$$\frac{\Gamma \vdash d : C(a, 1_a)}{\Gamma \vdash \mathsf{J}'\ d : (\Pi(x,\alpha) : S_a) C(x,\alpha)}$$

$$\mathsf{J}'\ d\ (a, 1_a) \equiv d : C(a, 1_a)$$

where S_a is the type $(\Sigma x : A)\ a =_A x$

We use the symbol $\equiv$ for definitional equality

Equality in Type Theory?

Principle 1: **Vacuum Cleaner Power Cord Principle**

Any element of $(\Sigma x : A)\ \mathsf{Id}_A\ a\ x$ is a pair x, α with $\alpha : a =_A x$.

In this type, there is a special element $a, 1_a$

The principle states that any element x, α is equal to this special element $a, 1_a$

Equality in Type Theory?

Principle 2: **Transport Principle**

If we have $\mathsf{Id}_T\ u\ v$ then $P(u) \to P(v)$

We write $\omega \cdot p : P(v)$ for $p : P(u)$ and $\omega : u =_T v$

Principle 3: we have $1 \cdot c \equiv c$

Equality in Type Theory

Example 1: Groupoid structure
If $y = z$ and $x = y$ then $x = z$
We write $\alpha.\beta : \; x = z$ for $\alpha : \; x = y$ and $\beta : \; y = z$
We have $\alpha.1 \equiv \alpha$

Using the vacuum cleaner cord principle we prove associativity, unit laws, and define inverse and prove the inverse laws

Equality in Type Theory

Any function $f : A \to B$ can be seen as a functor between the associated groupoid

We write $f\ \omega : \mathsf{Id}_B\ (f\ a_0)\ (f\ a_1)$ if $\omega : \mathsf{Id}_A\ a_0\ a_1$

We have $f\ 1 \equiv 1$

All the functor laws follow from the vacuum cleaner cord principle

If $f : (\Pi x : A) C(x)$ then for any $\omega : \mathsf{Id}_A\ a_0\ a_1$ we have $\omega \cdot (f\ a_0) = f\ a_1$

Equality in Type Theory

Example 2: assume that we have $\varphi : (\Pi x : A)\mathsf{Id}_A\ x\ a$ then

$$(\Pi\ x\ u : A)(\Pi\ \omega : \mathsf{Id}_A\ x\ u)\ \ \omega.(\varphi\ u) = \varphi\ x$$

This follows from $1.(\varphi\ a) = \varphi\ a$. In particular $\varphi\ a = 1$

Corollary: any two elements in $(\Pi x : A)\ x = a$ are equal if $a : A$ for any type A

If we have $\varphi, \psi : (\Pi x : A)x = a$ then for any x we have $(\psi\ x).(\varphi\ a) = \varphi\ x$ and hence $\psi\ x = \varphi\ x$

Equality in Type Theory

Example 3: Equality in a sum type $S = (\Sigma x : A)B$

We have $(a_0, b_0) =_S (a_1, b_1)$ iff

$$(\Sigma\alpha : a_0 =_A a_1)\ \alpha \cdot b_0 =_{B(a_1)} b_1$$

Notice that $(a, b) =_S (a, b')$ does not imply $b =_{B(a)} b'$!

Equality in Type Theory

Example 4: We say that $f : A \to B$ is a retraction if there exists $g : B \to A$ such that $x =_A g\ (f\ x)$ for all $x : A$

If $f : A \to B$ then for all $a_0\ a_1 : A$ the application

$$a_0 =_A a_1 \to f\ a_0 =_B f\ a_1$$

is a retraction

Equality in Type Theory

This strongly suggests a connection with algebraic topology and the notion of fibrations

Given a map $p : E \to B$ between topological space, view this map as defining a family of spaces $p^{-1}(b)$, the *fibers*, varying with $b : B$

This map is called a *Hurewicz fibration* iff any path $\sigma : I \to B$ and point $e : E$ such that $p(e) = \sigma(0)$ can be lifted to a path $\tilde{\sigma} : I \to E$ such that $\tilde{\sigma}(0) = e$ and $p \circ \tilde{\sigma} = \sigma : I \to B$

This is called a *regular* fibration if a constant path is lifted to a constant path.

Contractible types

Definition of contractibility contr A is defined to be

$$(\Sigma a : A)\ (\Pi x : A) x = a$$

The type $(\Sigma x : T) x = t$ is *always* contractible for any $t : T$ even if T is intuitively "very large"

Contractible types

If we have contr A then contr $(\mathrm{Id}_A\ a_0\ a_1)$ for any $a_0\ a_1 : A$

Space of path

(J.P.Serre) *when I was working on homotopy groups (around 1950), I convinced myself that, for a space X, there should exist a fiber space E, with base X, which is* contractible*; such a space would allow me (using Leray's methods) to do lots of computations on homotopy groups... But how to find it? It took me several weeks (a very long time, at the age I was then) to realize that the* space of "paths" *on X had all the necessary properties-if only I dared call it a "fiber space". This was the starting point of the loop space method in algebraic topology.*

Propositions

If we have $\alpha_{x,y} : x =_A y$ for all x and y in A then we have

$$(\Pi\omega : x =_A y)\ \omega.\alpha_{y,y} = \alpha_{x,y}$$

and hence

$$(\Pi\omega : x =_A y)\ \omega = \alpha_{x,y}$$

We say that A is a *proposition* iff $x =_A y$ for all x, y in A iff

$$(\Pi x : A)(\Pi y : A)\ \mathrm{contr}\ (x =_A y)$$

This is equivalent to

$$(\Pi x : A)(\Pi y : A)\ x =_A y$$

$\neg A$ is *always* a proposition, for any type A

Sets

We say that A is a *set* iff $x =_A y$ is a proposition for all x, y in A

N_2 is a set

Hedberg's Theorem: *If A has a decidable equality then A is a set*

Sets

We prove something stronger: if we have $\neg\neg(x =_A y) \;\rightarrow\; x =_A y$ i.e. the equality is stable, then A is a set

Lemma: *If we have an operation* $f\ \omega : x =_A y$ *for* $\omega : x =_A y$ *then* $f\ \omega = (f\ 1).\omega$ *for any* $\omega : x =_A y$

This follows from the vacuum cleaner cord principle

If the equality is stable then we have $f\ \omega = f\ \omega'$ for any ω and ω' of type $x =_A y$ and hence $\omega = \omega'$

Sets

N_0 is a proposition

N_1 is contractible

$N_2,\ N$ are sets and not propositions

With one universe we can show $\neg(0 = 1)$

To be a set is the same as to satisfy the uniqueness of identity proofs

In general $A + B$ is a set if A, B are propositions, however prop $(A + \neg A)$ if prop A

Voevodsky's stratification of types

contractible, proposition, set

hlevel 0 A is defined to be contr A

hlevel $(n+1)$ A is defined to be $(\Pi x : A)(\Pi x' : A)$ hlevel n $(x =_A y)$

Voevodsky's stratification of types

level 1 propositions

level 2 sets

level 3 groupoids

Voevodsky's stratification of types

If A is contractible then each equality type $a_0 =_A a_1$ is also contractible

Definition of equivalence

The notion of equivalence captures in an uniform way: logical equivalence, isomorphisms (between sets), equivalence (between groupoids), ...

Let f be a function $A \to B$

The type $F(y) = (\Sigma x : A)\ f\ x = y$ is the *fiber* of f at y

$f : A \to B$ is an *equivalence* iff

$$(\Pi y : B)\mathrm{contr}\ F(y)$$

An application is an equivalence iff all fibers are contractible

Define IsEquiv f to be $(\Pi y : B)\mathrm{contr}\ F(y)$

Equality in the fibers

We have $(x, \omega) =_{F(y)} (x', \omega')$ iff $(\Sigma\alpha : x =_A x') \quad (f\ \alpha).\omega' = \omega$

Equivalence

So $f : A \to B$ is an equivalence iff we have $g : B \to A$ and $\psi_y : f\ (g\ y) =_B y$ and if $\omega : f\ x =_B y$ we have $\omega^* : x =_A g\ y$ such that $(f\ \omega^*).\psi_y = \omega$

In particular we have $\alpha_x : x =_A g\ (f\ x)$ such that $(f\ \alpha_x).\psi_{f\ x} = 1$

Axiom of Univalence

Let $\alpha_{A,B}$ *be the canonical application* $\mathrm{Id}_U\ A\ B \to \mathrm{Equiv}\ A\ B$ *then we have* $\mathrm{IsEquiv}\ \alpha_{A,B}$

$$A = B \ \simeq \ (A \simeq B)$$

Axiom of Univalence

In particular we have $A \simeq B \leftrightarrow A = B$ but the axiom is much more subtle

Equivalence

$f : A \to B$ is a homotopy equivalence iff there exists $g : B \to A$ such that $x =_A g\ (f\ x)$ and $f\ (g\ y) =_B y$ for all $x : A$ and $y : B$

Graduate Lemma: *If f is a homotopy equivalence then it is an equivalence*

Proof at the blackboard (no simple direct proof)

Applications of the Graduate Lemma

The canonical maps

$$(\Sigma x : A)(B(x) + C(x)) \to (\Sigma x : A)B(x) + (\Sigma x : A)C(x)$$

$$(\Pi x : A)(\Sigma y : B(x))C(x, y) \to (\Sigma f : (\Pi x : A)B(x))(\Pi x : A)C(x, f\ x)$$

$$(a_0, b_0) =_{(\Sigma x:A)B} (a_1, b_1) \to (\Sigma \alpha : a_0 =_A a_1)\alpha \cdot b_0 =_{B(a_1)} b_1$$

are all equivalences

Applications of the Graduate Lemma

The application $\neg : N_2 \to N_2$ is an equivalence

Indeed we have $\neg\ (\neg x) =_{N_2} x$ for all x in N_2

It follows that $\mathrm{Path}_{U_0}\ N_2\ N_2$ is not a proposition and hence U_0 is not a set!

Notice that by univalence we have $(\mathrm{Path}_{U_0}\ N_2\ N2) =_{U_1} \mathrm{Equiv}\ N_2\ N_2$ and we have two distinct elements in $\mathrm{Equiv}\ N_2\ N_2$

$\mathrm{Equiv}\ N_2\ N_2 =_{U_0} N_2$

Extensionality

The axiom of univalence generalizes one extensionality axiom of Church's simple type theory stating that two equivalent propositions are equal

Extensionality for functions? There are various possible statements

Theorem: *The following statements are* logically *equivalent*

1. $((\Pi x : A) f\ x =_{B(x)} g\ x) \to f = g$
2. $((\Pi x : A)\mathrm{contr}\ B(x)) \to \mathrm{contr}\ (\Pi x : A) B(x)$
3. *the canonical map* $f = g \to (\Pi x : A) f\ x =_{B(x)} g\ x$ *is an equivalence*

Extensionality

Lemma: *if we have a family of maps* $\varphi_x : B(x) \to C(x)$ *and the total map* $(x, b) \longmapsto (x, \varphi_x\ b)$ *is an equivalence then each map* φ_x *is an equivalence*

This is a standard lemma in the theory of fibrations in homotopy theory

It follows from the fact that the fibers of the total map at a point (x_0, c_0) is equivalent to the fiber of the map φ_{x_0} at the point c_0

Propositions

It follows from function extensionality that a product of a family of types of level n is itself of level n

In particular prop $(\Pi x : A)B(x)$ if we have $(\Pi x : A)\text{prop } B(x)$ for *any* type A

A product of propositions is always a proposition

Properties

We have prop $(\mathsf{contr} A)$ for any type A

It follows from function extensionality that the axiom of univalence and the axiom of extensionality (second and third form) are themselves propositions

If we have $(\Pi x : A)\mathsf{prop}\ B(x)$ then the canonical map $\mathsf{p} : (\Sigma x : A)B(x) \rightarrow A$ satisfies

$$\mathsf{p}\ c = \mathsf{p}\ c' \rightarrow c = c'$$

Isomorphic structures are equal

Representation of (small) monoids

$$(\Sigma A : U_0)(\Sigma h : \text{set } A)(\Sigma e : A)(\Sigma f : A \to A \to A) \ldots$$

where $\ldots$ are the axioms of monoids, conjunction of statements such as

$$(\Pi x : A)\ f\ \ e\ x =_A x$$

Notice that all these equational conditions are *proposition* since A is assumed to be a *set*

Isomorphic structures are equal

In general a *structure* will be defined by $T(A)$ $(A : U_0)$ and $P(A, x)$ $(A : U_0, x : T(A))$ with e.g.

$$T(A) = A \times (A \to A \to A)$$

such that prop $P(A, x)$ if $A : U_0$ and $x : T(A)$

We can define for each equivalence $f : A \to B$ an equivalence $T(f) : T(A) \to T(B)$ and we have $T(1) = 1$

Transport of structures

If $f : A \to B$ and $x : T(A)$ is a T-structure on A then $T(f)\ x$ is a T-structure on B

We say that (A, x) and (B, y) are *isomorphic* iff there exists an equivalence $f : A \to B$ such that $T(f)\ x =_{T(B)} y$

By univalence we can write $f = \alpha_{A,B}\ p$ with $p : A =_U B$

We have $T(\alpha_{A,B}\ p)\ x = p \cdot x$ for any $x : T(A)$ and any $p : A =_{U_0} B$ by the vaccum cleaner cord principle

If $T(f)\ x = y$ we have $p : A =_{U_0} B$ and $p \cdot x = y$
Hence $(A, x) =_S (B, y)$ where $S = (\Sigma X : U_0) T(X)$
This shows that *isomorphic* structures are *equal*

No global choice for sets

There is no global choice function

$$\neg((\Pi X : U)\ \mathsf{set}\ X\ \to \neg\ (\neg\ X)\ \to X)$$

Indeed any such choice operation has to be uniform w.r.t. all automorphisms, and this is impossible