Logic, Automata, and Games IV: Decidability of Monadic Theories

Wolfgang Thomas

RWTH AACHEN

Nordic School, Nordfjordeid, May 2013

Overview

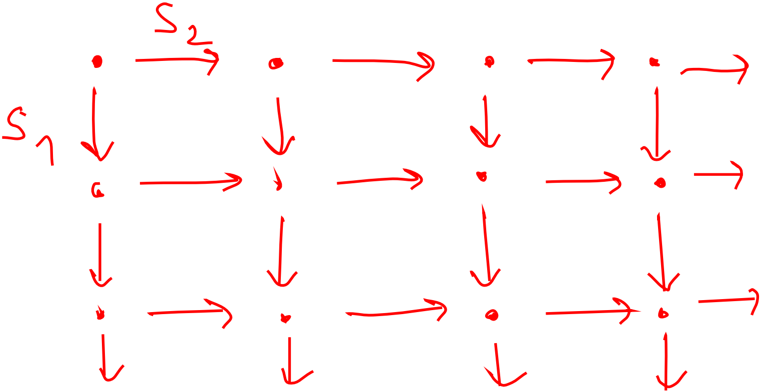
1. **Undecidability Results**
2. **Decidability Results**
3. **The Pushdown Hierarchy**

The Infinite Grid

The infinite grid is the structure

$$G_2 = (\mathbb{N} \times \mathbb{N}, (0,0), S_1, S_2)$$

where $S_1(i,j) = (i+1, j), \quad S_2(i,j) = (i, j+1)$



Undecidability of Monadic Grid-Theory

The monadic second-order theory of the infinite grid is undecidable.

Proof

by reduction of the halting problem for Turing machines:

For any TM M construct a sentence φ_M of the monadic second-order language of G_2 such that

M halts when started on the empty tape iff $G_2 \models \varphi_M$.

Configurations of M

Assume that M works on a left-bounded tape.

A halting computation of M can be coded by a finite sequence of configuration words

$C_0, C_1, \ldots, C_m$.

We can arrange the configurations row by row in a right-infinite rectangular array:

$$\begin{array}{cccccccc} q_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & \ldots \\ a_1 & q_1 & a_0 & a_0 & a_0 & a_0 & a_0 & \ldots \\ q_0 & a_1 & a_2 & a_0 & a_0 & a_0 & a_0 & \ldots \\ a_3 & q_2 & a_2 & a_0 & a_0 & a_0 & a_0 & \ldots \end{array}$$

etc.

Describing an M-Run

The sentence φ_M will express over G_2 the existence of such an array of configurations.

$a_0, \ldots, a_n$ **are the tape symbols (**a_0 **is the blank)**
$q_0, \ldots, q_k$ **are the states of** M**, special halting state** q_s
We use set variables $X_0, \ldots, X_n, Y_0, \ldots, Y_k$
X_i **collects the grid positions where** a_i **occurs,**
Y_i **collects the grid positions where state** q_i **occurs.**
φ_M **:** $\exists X_0, \ldots, X_n, Y_0, \ldots, Y_k$ $($**Partition**$(X_0, \ldots, Y_k)$

- $\wedge$ **"the first row is the initial** M**-configuration"**
- $\wedge$ **"a successor row is the successor configuration of the preceding one"**
- $\wedge$ **"at some position the halting state is reached"**$)$

RWTHAACHEN

Use of Interpretations

An MSO-interpretation of a structure $\mathcal{A} = (A, R^{\mathcal{A}}, \ldots)$ in a structure $\mathcal{B}$ is a description of $\mathcal{A}$ in $\mathcal{B}$

Here we use MSO for the description.

Assume $\mathcal{A}$ is MSO-interpretable in $\mathcal{B}$.

Then:

MTh$(\mathcal{A})$ undecidable implies MTh$(\mathcal{B})$ undecidable.

MTh$(\mathcal{B})$ decidable implies MTh$(\mathcal{A})$ decidable.

Interpretations Formally

An MSO-interpretation of a structure $\mathcal{A} = (A, R^{\mathcal{A}}, \ldots)$ in a structure $\mathcal{B}$ is given by

- **a “domain formula” $\varphi(x)$**
- **for each relation $R^{\mathcal{A}}$ of $\mathcal{A}$, say of arity m, an MSO-formula $\psi(x_1, \ldots, x_m)$**

such that $\mathcal{A}$ is isomorphic to $(\varphi^{\mathcal{B}}, \psi^{\mathcal{B}}, \ldots)$

Then there is a transformation of MSO-sentences χ (in the signature of $\mathcal{A}$) to sentences χ' (in the signature of $\mathcal{B}$) such that
$\mathcal{A} \models \chi$ iff $\mathcal{B} \models \chi'$.

Consequence:

If $\mathcal{A}$ is MSO-interpretable in $\mathcal{B}$ and the MSO-theory of $\mathcal{B}$ is decidable, then so is the MSO-theory of $\mathcal{A}$.

A Hidden Grid

Consider the expansion of the tree T_2 by the two first-letter-adding functions:

$$p_0(w) = 0 \cdot w, \quad p_1(w) = 1 \cdot w$$

The MSO-theory of (T_2, p_0, p_1) is undecidable.

Proof: Give interpretation of G_2 in (T_2, p_0, p_1)

Domain formula, using $\sigma_i(z, z') : zi = z' \;\; (i = 0, 1)$

$$\varphi(x) : \exists y(\sigma_0^*(\varepsilon, y) \wedge \sigma_1^*(y, x))$$

$$\psi_1(x, y) : p_0(x) = y, \;\; \psi_2(x, y) : x1 = y$$

Another Hidden Grid

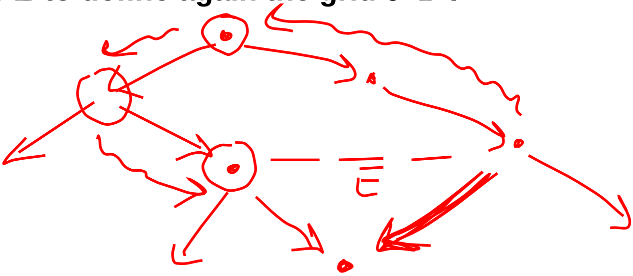
Consider the binary tree with Equal-Level Predicate E

$$E(u,v) \quad :\Leftrightarrow \quad |u| = |v|$$

Obtain (T_2, E)**.**

The MSO-theory of (T_2, E) **is undecidable.**

Proof: Use E **to define again the grid** 0^*1^***.**



Quantification over Binary Relations

By the results of Gödel, Tarski, Turing we know:

The first-order theory of $(\mathbb{N}, +, \cdot, 0, 1)$ is undecidable.

Already Gödel remarked in 1931:

In the second-order language (with quantifiers over elements and relations) one can define define $+$ and $\cdot$ in $(\mathbb{N}, +1)$.

Consequence:

The second-order theory of $(\mathbb{N} + 1)$ is undecidable.

$x + y = z$

iff

$\forall R([R(0, x) \wedge \forall s, t(R(s, t) \rightarrow R(s + 1, t + 1))] \rightarrow R(y, z))$

Adding Double Function to $(\mathbb{N}, +1)$

$\text{double}(x) := 2x$.

Robinson 1958:

The (weak) MSO-theory of $(\mathbb{N}, +1, \text{double})$ is undecidable.

We follow a proof idea of Elgot and Rabin [JSL 31 (1966)].

Code a relation $R = \{(m_1, n_1), \ldots, (m_k, n_k)\}$

by a set $M_R = \{m'_1 < n'_1 < \ldots < m'_k < n'_k\}$

For each n **we need an infinite set of code numbers.**

Take as codes of n **all numbers** $2^i \cdot (\text{double}(n) + 1)$

Example

$$R = \{(2,1),(0,2)\}$$

A code set M_R contains

$$1 \cdot 5 < 2 \cdot 3 < 8 \cdot 1 < 2 \cdot 5$$

A Remark

There is an MSO-formula $\mathrm{OddPos}(X, x)$ that expresses

- $X(x)$
- **in the $<$-listing of X-elements, x occurs on an odd position.**

Use $\psi(X, z, z')$:

$X(z) \wedge X(z')$

$\wedge$ **there is precisely one y between z, z' with $X(y)$**

$\mathrm{OddPos}(X, x)$ **:** $\psi^*(X, \min(X), x)$

$\mathrm{Next}(X, x, y)$ **says "in X, y is the next element after x**

Definability of Decoding

Let $\varphi_2(z, z') := \ \mathbf{double}(z) = z'$

Then

"s is a code of x": $\exists y(\mathbf{double}(x) + 1 = y \wedge \varphi_2^*(y, s))$

Translation of $\exists R(R(x, y) \ldots)$**:**

$$\exists X(\exists s \exists t(s \text{ is code of } x \wedge t \text{ is code of } y$$
$$\wedge \mathbf{OddPos}(X, s) \wedge \mathbf{Next}(X, s, t))$$

A Sharper Result

Let $f : \mathbb{N} \to \mathbb{N}$ be

- **strictly increasing,**
- **$f - \mathrm{id}_{\mathbb{N}}$ be monotone and unbounded.**

Then MTh$(\mathbb{N}, +1, 0, f)$ is undecidable.

[W. Th., A note on undecidable extensions of monadic second order arithmetic, Arch math. Logik 17 (1975)]

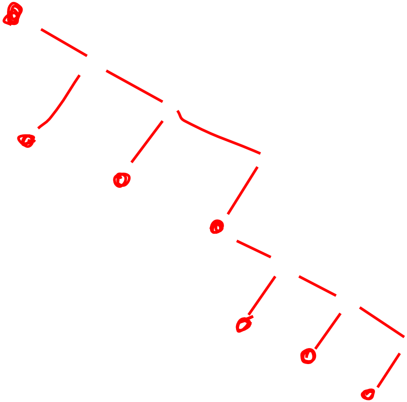
Decidability Results

A First Example

Show Rabin's Tree Theorem for $T_3 = (\{0,1,2\}^*, S_0^3, S_1^3, S_2^3)$.

Idea: Obtain a copy of T_3 in T_2:

Consider T_2-vertices in $T = (10 + 110 + 1110)^*$.



Interpretation: Details

The element $i_1 \ldots i_m$ of T_3 is coded by

$1^{i_1+1}0 \ldots 1^{i_m+1}0$ in T_2.

Define the set of codes by

$\varphi(x)$: "x is in the closure of ε under 10-, 110-, and 1110-successors"

Define the 0-th, 1-st 2-nd successors by $\psi_0(x,y), \psi_1(x,y), \psi_2(x,y)$

The structure $(\varphi^{T_2}, (\psi_i^{T_2})_{i=0,1,2}, \varepsilon)$ is isomorphic to T_3.

Another Interpretation

MTh$(\mathbb{Q}, <)$ is decidable. (Rabin 1969)

Work with the tree nodes $w01$

With the lexicographic order they give a countable dense linear order.

This order is isomorphic to $(\mathbb{Q}, <)$ (Cantor)

So we have an interpretation of $(\mathbb{Q}, <)$ in T_2.

Much more difficult: **MTh$(\mathbb{R}, <)$ is undecidable.** (Shelah 1975)

Pushdown Graphs

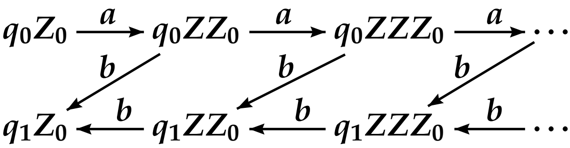
Consider $\mathcal{A}$ for language $L = \{a^n b^n \mid n \geq 0\}$:

$\mathcal{A} = (\{q_0, q_1\}, \{a, b\}, \{Z_0, Z\}, q_0, Z_0, \Delta)$ **with**

$$\Delta = \left\{ \begin{array}{ll} (q_0, Z_0, a, q_0, ZZ_0), & (q_0, Z, a, q_0, ZZ), \\ (q_0, Z, b, q_1, \varepsilon), & (q_1, Z, b, q_1, \varepsilon) \end{array} \right\}$$

Initial and final configuration: $q_0 Z_0$

The associated pushdown graph (of reachable configurations only) is:



Interpretation: Second Example

A pushdown graph is MSO-interpretable in T_2

Given pushdown automaton $\mathcal{A}$ with stack alphabet $\{1, \ldots, k\}$ and states $q_1, \ldots, q_m$.

Let $G_{\mathcal{A}} = (V_{\mathcal{A}}, E_{\mathcal{A}})$ be the corresponding PD graph.
$n := \max\{k, m\}$

Find an MSO-interpretation of $G_{\mathcal{A}}$ in T_n.

Represent configuration $(q_j, i_1 \ldots i_r)$ by the vertex $i_r \ldots i_1 j$.

$\mathcal{A}$-steps lead to local moves in T_n.

E.g. a push step from vertex $i_r \ldots i_1 j$ to $i_r \ldots i_1 i_0 j'$.

These edges are easily definable in MSO.

Hence: **The MSO-theory of a PD graph is decidable.**

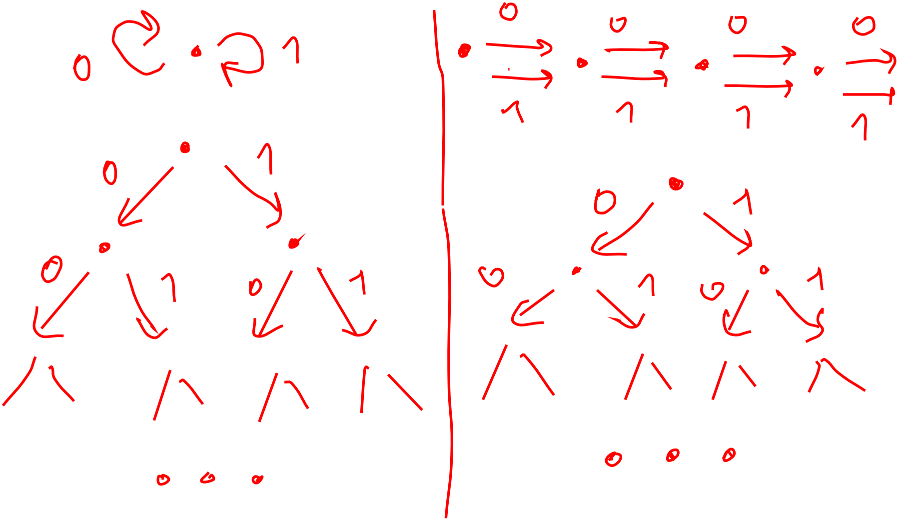
Unfoldings

Given a graph $(V, (E_a)_{a\in\Sigma}, (P_b)_{b\in\Sigma'})$

the unfolding of G **from a given vertex** v_0 **is the following tree** $T_G(v_0) = (V', (E'_a)_{a\in\Sigma}, (P'_b)_{b\in\Sigma'})$**:**

- V' **consists of the vertices** $v_0 a_1 v_1 \dots a_r v_r$ **with** $(v_{i-1}, v_i) \in E_{a_i}$,
- E'_a **contains the pairs** $(v_0 a_1 v_1 \dots a_r v_r, v_0 a_1 v_1 \dots a_r v_r a v)$ **with** $(v_r, v) \in E_a$,
- P'_b **the vertices** $v_0 a_1 v_1 \dots a_r v_r$ **with** $v_r \in P_b$.

Examples



Unfolding Preserves Decidability

Theorem (Muchnik, Courcelle/Walukiewicz)

If the MSO-theory of G is decidable and v_0 is an MSO-definable vertex of G, then the MSO-theory of $T_G(v_0)$ is decidable.

Proof Architecture (for Pushdown Graphs)

Given an unfolding T of a pushdown graph G.

T is finitely branching, with labels say in Σ inherited from G.

For each MSO-formula $\varphi(X_1, \ldots, X_n)$ find a parity tree automaton $\mathcal{A}_\varphi$ such that

$\mathcal{A}_\varphi$ accepts $T(P_1, \ldots, P_n)$ iff $T[P_1, \ldots, P_n) \models \varphi(X_1, \ldots, X_n)$

The construction of the $\mathcal{A}\varphi$ follows precisely the pattern of Rabin's equivalence theorem.

Essential: In the complementation step we use the finite out-degree of G.

The general case is more involved.

Caucal's Proposal

We have now two processes which preserve decidability of MSO-theory:

- interpretation (transforming a tree into a graph)
- unfolding (transforming a graph into a tree)

Let us apply them in alternation!

We obtain the Caucal hierarchy or pushdown hierarchy.

Definition

- $\mathcal{T}_0 =$ the class of finite trees
- $\mathcal{G}_n =$ the class of graphs which are MSO-interpretable in a tree of $\mathcal{T}_n$
- $\mathcal{T}_{n+1} =$ the class of unfoldings of graphs in G_n

Each structure in the pushdown hierarchy has a decidable MSO-theory.

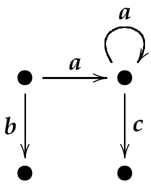
Nontrivial fact:

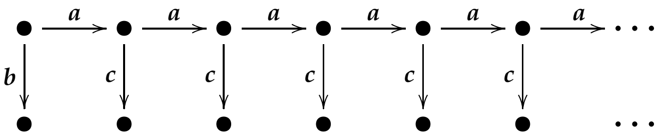
The sequence $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$ is strictly increasing.

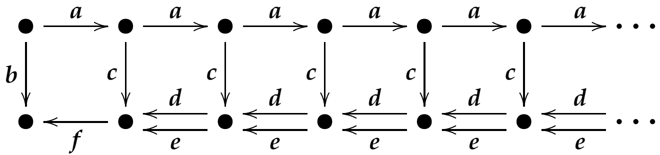
The First Levels

- $\mathcal{G}_0$ is the class of finite graphs.
- $\mathcal{T}_1$ contains the regular trees.
- $\mathcal{G}_1$ contains the prefix-recognizable graphs.

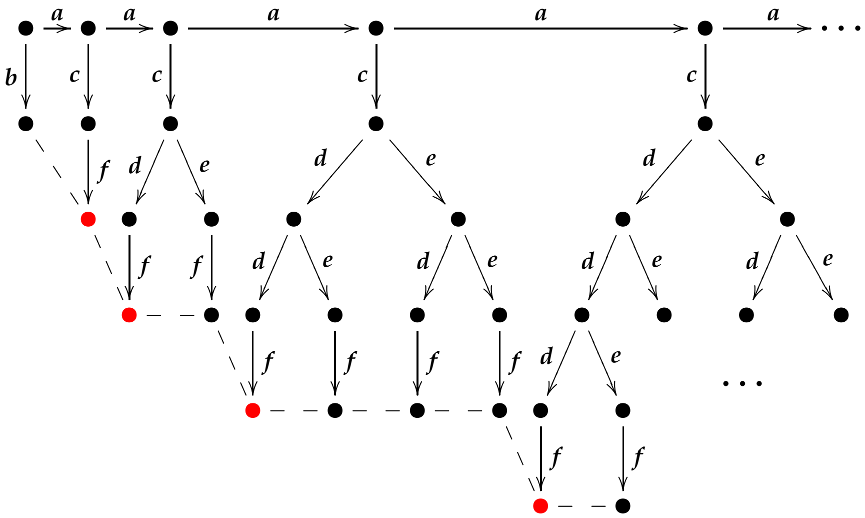
A Finite Graph, a Regular Tree, a PD Graph







Unfolding Again



Interpretation of Bottom Line

The sequence of leaves defines a copy of the successor structure of the natural numbers.

We give interpretation with regular expressions rather than MSO

Domain expression: $b + a^* c(d + e)^* f$

Successor relation:

$\overline{b}acf +$

$\overline{f}\overline{e}^*\overline{c}acd^* f +$

$\overline{f}\overline{e}^*\overline{d}ed^* f$

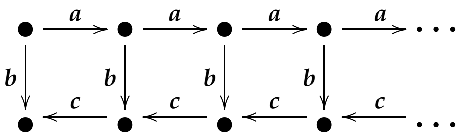
Predicate "power of 2": $b + a^* cd^* f$

Result: $(\mathbb{N}, \mathrm{Succ}, \mathrm{Pow}_2)$ **is a structure in the Caucal hierarchy.**

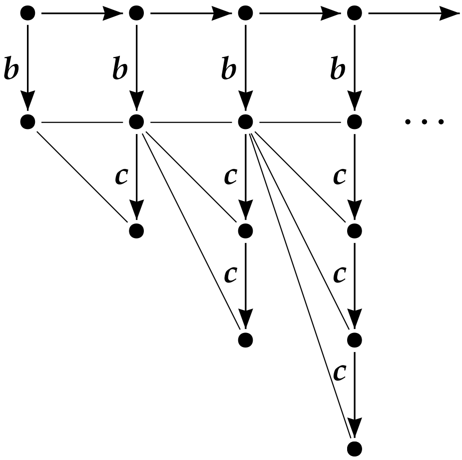
Towards Factorial Predicate

$(\mathbb{N}, \mathbf{Succ}, \mathbf{Fac})$

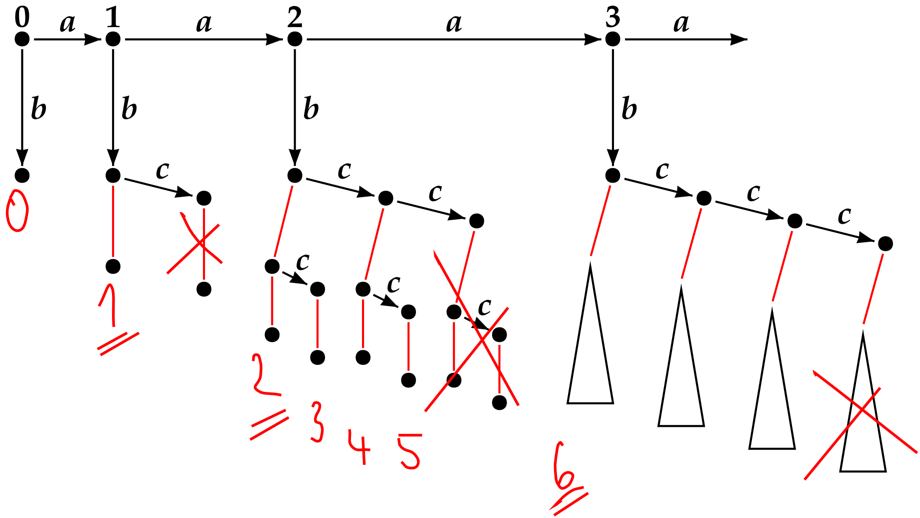
We start as follows:



Continuation: Unfolding and Interpretation



Obtaining $(\mathbb{N}, +1, , \mathrm{Fac})$



Scope of Hierarchy?

The pushdown hierarchy is a very rich class of structures all of which have a decidable MSO-theory.

Open questions:

- **Understand which structures belong to the hierarchy**
- **Compute the smallest level on which a structure occurs**

There are structures $\mathcal{S}$ which have a decidable monadic theory but do not belong to the hierarchy.

(Example: Consider the set P of iterated 2-powers 1, 2, 2^2, 2^{2^2}, etc., and take $(\mathbb{N}, +1, P)$.)