Logic, Automata, and Games IV: Decidability of Monadic Theories

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Overview

1. Undecidability Results
2. Decidability Results
3. The Pushdown Hierarchy
The infinite grid is the structure

\[ G_2 = (\mathbb{N} \times \mathbb{N}, (0, 0), S_1, S_2) \]

where \( S_1(i, j) = (i + 1, j) \), \( S_2(i, j) = (i, j + 1) \)
Undecidability of Monadic Grid-Theory

The monadic second-order theory of the infinite grid is undecidable.

Proof

by reduction of the halting problem for Turing machines:

For any TM $M$ construct a sentence $\varphi_M$ of the monadic second-order language of $G_2$ such that

$M$ halts when started on the empty tape iff $G_2 \models \varphi_M$. 
Configurations of $M$

Assume that $M$ works on a left-bounded tape.

A halting computation of $M$ can be coded by a finite sequence of configuration words $C_0, C_1, \ldots, C_m$.

We can arrange the configurations row by row in a right-infinite rectangular array:

\[
\begin{array}{cccccccc}
q_0 & a_0 & a_0 & a_0 & a_0 & a_0 & a_0 & \ldots \\
a_1 & q_1 & a_0 & a_0 & a_0 & a_0 & a_0 & \ldots \\
q_0 & a_1 & a_2 & a_0 & a_0 & a_0 & a_0 & \ldots \\
a_3 & q_2 & a_2 & a_0 & a_0 & a_0 & a_0 & \ldots \\
\end{array}
\]

etc.
Describing an $M$-Run

The sentence $\varphi_M$ will express over $G_2$ the existence of such an array of configurations.

$a_0, \ldots, a_n$ are the tape symbols ($a_0$ is the blank)

$q_0, \ldots, q_k$ are the states of $M$, special halting state $q_s$

We use set variables $X_0, \ldots, X_n, Y_0, \ldots, Y_k$

$X_i$ collects the grid positions where $a_i$ occurs,

$Y_i$ collects the grid positions where state $q_i$ occurs.

$\varphi_M : \exists X_0, \ldots, X_n, Y_0, \ldots, Y_k \left( \text{Partition}(X_0, \ldots, Y_k) \right)$

$\land$ “the first row is the initial $M$-configuration”

$\land$ “a successor row is the successor configuration of the preceding one”

$\land$ “at some position the halting state is reached”
Use of Interpretations

An MSO-interpretation of a structure $\mathcal{A} = (A, R^A, \ldots)$ in a structure $\mathcal{B}$ is a description of $\mathcal{A}$ in $\mathcal{B}$

Here we use MSO for the description.

Assume $\mathcal{A}$ is MSO-interpretatable in $\mathcal{B}$.

Then:

$\text{MTh}(\mathcal{A})$ undecidable implies $\text{MTh}(\mathcal{B})$ undecidable.

$\text{MTh}(\mathcal{B})$ decidable implies $\text{MTh}(\mathcal{A})$ decidable.
Interpretations Formally

An MSO-interpretation of a structure $\mathcal{A} = (A, R^A, \ldots)$ in a structure $\mathcal{B}$ is given by

- a “domain formula” $\varphi(x)$
- for each relation $R^A$ of $\mathcal{A}$, say of arity $m$, an MSO-formula $\psi(x_1, \ldots, x_m)$

such that $\mathcal{A}$ is isomorphic to $(\varphi^B, \psi^B, \ldots)$

Then there is a transformation of MSO-sentences $\chi$ (in the signature of $\mathcal{A}$) to sentences $\chi'$ (in the signature of $\mathcal{B}$) such that

$\mathcal{A} \models \chi$ iff $\mathcal{B} \models \chi'$.

Consequence:

If $\mathcal{A}$ is MSO-interpretable in $\mathcal{B}$ and the MSO-theory of $\mathcal{B}$ is decidable, then so is the MSO-theory of $\mathcal{A}$.
A Hidden Grid

Consider the expansion of the tree $T_2$ by the two first-letter-adding functions:

$$p_0(w) = 0 \cdot w, \quad p_1(w) = 1 \cdot w$$

The MSO-theory of $(T_2, p_0, p_1)$ is undecidable.

Proof: Give interpretation of $G_2$ in $(T_2, p_0, p_1)$

Domain formula, using $\sigma_i(z, z') : zi = z'$ ($i = 0, 1$)

$$\varphi(x) : \exists y (\sigma_0^*(\varepsilon, y) \land \sigma_1^*(y, x))$$

$$\psi_1(x, y) : p_0(x) = y, \quad \psi_2(x, y) : x1 = y$$
Another Hidden Grid

Consider the binary tree with Equal-Level Predicate $E$

\[ E(u, v) \iff |u| = |v| \]

Obtain $(T_2, E)$.

The MSO-theory of $(T_2, E)$ is undecidable.

Proof: Use $E$ to define again the grid $0^*1^*$. 

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Quantification over Binary Relations

By the results of Gödel, Tarski, Turing we know:

The first-order theory of \((\mathbb{N}, +, \cdot, 0, 1)\) is undecidable.

Already Gödel remarked in 1931:

In the second-order language (with quantifiers over elements and relations) one can define define \(+\) and \(\cdot\) in \((\mathbb{N}, + 1)\).

Consequence:

The second-order theory of \((\mathbb{N} + 1)\) is undecidable.

\(x + y = z\)

iff

\[\forall R([R(0, x) \land \forall s, t(R(s, t) \rightarrow R(s + 1, t + 1))] \rightarrow R(y, z))]\]
double\( (x) := 2x \).

Robinson 1958:

The (weak) MSO-theory of \( (\mathbb{N}, +1, \text{double}) \) is undecidable.

We follow a proof idea of Elgot and Rabin [JSL 31 (1966)].

Code a relation \( R = \{(m_1, n_1), \ldots, (m_k, n_k)\} \)

by a set \( M_R = \{m'_1 < n'_1 < \ldots < m'_k < n'_k\} \)

For each \( n \) we need an infinite set of code numbers.

Take as codes of \( n \) all numbers \( 2^i \cdot (\text{double}(n) + 1) \)
Example

\[ R = \{(2, 1), (0, 2)\} \]

A code set \(M_R\) contains

\[ 1 \cdot 5 < 2 \cdot 3 < 8 \cdot 1 < 2 \cdot 5 \]
There is an MSO-formula $\text{OddPos}(X,x)$ that expresses

- $X(x)$
- in the $\prec$-listing of $X$-elements, $x$ occurs on an odd position.

Use $\psi(X,z,z')$:

$X(z) \land X(z')$

$\land$ there is precisely one $y$ between $z, z'$ with $X(y)$

$\text{OddPos}(X,x) : \psi^*(X, \min(X), x)$

$\text{Next}(X,x,y)$ says “in $X$, $y$ is the next element after $x$"
Definability of Decoding

Let \( \varphi_2(z, z') := \text{double}(z) = z' \)

Then

“\( s \) is a code of \( x \)”: \( \exists y (\text{double}(x) + 1 = y \land \varphi^*_2(y, s)) \)

Translation of \( \exists R(R(x, y) \ldots) \):

\( \exists X(\exists s \exists t (s \text{ is code of } x \land t \text{ is code of } y \land \text{OddPos}(X, s) \land \text{Next}(X, s, t)) \)
A Sharper Result

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be

- strictly increasing,
- $f - \text{id}_\mathbb{N}$ be monotone and unbounded.

Then $\text{MTh}(\mathbb{N}, +1, 0, f)$ is undecidable.

[W. Th., A note on undecidable extensions of monadic second order arithmetic, Arch math. Logik 17 (1975)]
Decidability Results
A First Example

Show Rabin’s Tree Theorem for $T_3 = (\{0, 1, 2\}^*, S_0^3, S_1^3, S_2^3)$.

Idea: Obtain a copy of $T_3$ in $T_2$:

Consider $T_2$-vertices in $T = (10 + 110 + 1110)^*$. 
The element $i_1 \ldots i_m$ of $T_3$ is coded by
$1^{i_1+1}0 \ldots 1^{i_m+1}0$ in $T_2$.

Define the set of codes by

$\varphi(x)$: “$x$ is in the closure of $\varepsilon$ under 10-, 110-, and 1110-successors”

Define the 0-th, 1-st 2-nd successors by

$\psi_0(x, y), \psi_1(x, y), \psi_2(x, y)$

The structure $(\varphi^{T_2}_i, (\psi_i^{T_2})_{i=0,1,2}, \varepsilon)$ is isomorphic to $T_3$. 
Another Interpretation

$$\text{MTh}(\mathbb{Q}, <) \text{ is decidable.} \quad \text{(Rabin 1969)}$$

Work with the tree nodes $w_01$

With the lexicographic order they give a countable dense linear order.

This order is isomorphic to $(\mathbb{Q}, <)$ (Cantor)

So we have an interpretation of $(\mathbb{Q}, <)$ in $T_2$.

Much more difficult: $\text{MTh}(\mathbb{R}, <) \text{ is undecidable.} \quad \text{(Shelah 1975)}$
Pushdown Graphs

Consider $A$ for language $L = \{ a^n b^n \mid n \geq 0 \}$:

$A = (\{q_0, q_1\}, \{a, b\}, \{Z_0, Z\}, q_0, Z_0, \Delta)$ with

$$\Delta = \begin{cases} 
(q_0, Z_0, a, q_0, ZZ_0), & (q_0, Z, a, q_0, ZZ), \\
(q_0, Z, b, q_1, \varepsilon), & (q_1, Z, b, q_1, \varepsilon) 
\end{cases}$$

Initial and final configuration: $q_0Z_0$

The associated pushdown graph (of reachable configurations only) is:

$$q_0Z_0 \xrightarrow{a} q_0ZZ_0 \xrightarrow{a} q_0ZZZ_0 \xrightarrow{a} \ldots$$

$$b \quad b \quad b$$

$$q_1Z_0 \xleftarrow{b} q_1ZZ_0 \xleftarrow{b} q_1ZZZ_0 \xleftarrow{b} \ldots$$

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A pushdown graph is MSO-interpretable in $T_2$

Given pushdown automaton $\mathcal{A}$ with stack alphabet $\{1, \ldots, k\}$ and states $q_1, \ldots, q_m$.

Let $G_{\mathcal{A}} = (V_{\mathcal{A}}, E_{\mathcal{A}})$ be the corresponding PD graph.

$n := \max\{k, m\}$

Find an MSO-interpretation of $G_{\mathcal{A}}$ in $T_n$.

Represent configuration $(q_j, i_1 \ldots i_r)$ by the vertex $i_r \ldots i_1 j$.

$\mathcal{A}$-steps lead to local moves in $T_n$.

E.g. a push step from vertex $i_r \ldots i_1 j$ to $i_r \ldots i_1 i_0 j'$.

These edges are easily definable in MSO.

Hence: The MSO-theory of a PD graph is decidable.
Unfoldings

Given a graph \((V, (E_a)_{a \in \Sigma}, (P_b)_{b \in \Sigma'})\)

the unfolding of \(G\) from a given vertex \(v_0\) is the following tree

\[ T_G(v_0) = (V', (E'_a)_{a \in \Sigma}, (P'_b)_{b \in \Sigma'}) \updownarrow \]

- \(V'\) consists of the vertices \(v_0a_1v_1\ldots a_rv_r\) with \((v_{i-1}, v_i) \in E_a\),
- \(E'_a\) contains the pairs \((v_0a_1v_1\ldots a_rv_r, v_0a_1v_1\ldots a_rv_rav)\) with \((v_r, v) \in E_a\),
- \(P'_b\) the vertices \(v_0a_1v_1\ldots a_rv_r\) with \(v_r \in P_b\).
Unfolding Preserves Decidability

Theorem (Muchnik, Courcelle/Walukiewicz)

If the MSO-theory of $G$ is decidable and $v_0$ is an MSO-definable vertex of $G$, then the MSO-theory of $T_G(v_0)$ is decidable.
Proof Architecture (for Pushdown Graphs)

Given an unfolding $T$ of a pushdown graph $G$.

$T$ is finitely branching, with labels say in $\Sigma$ inherited from $G$.

For each MSO-formula $\varphi(X_1, \ldots, X_n)$ find a parity tree automaton $A_\varphi$ such that

$A_\varphi$ accepts $T(P_1, \ldots, P_n)$ iff $T[P_1, \ldots, P_n] \models \varphi(X_1, \ldots, X_n)$

The construction of the $A_\varphi$ follows precisely the pattern of Rabin's equivalence theorem.

Essential: In the complementation step we use the finite out-degree of $G$.

The general case is more involved.
Caucal’s Proposal

We have now two processes which preserve decidability of MSO-theory:

- interpretation (transforming a tree into a graph)
- unfolding (transforming a graph into a tree)

Let us apply them in alternation!

We obtain the Caucal hierarchy or pushdown hierarchy.
Definition

- $\mathcal{T}_0 =$ the class of finite trees
- $\mathcal{G}_n =$ the class of graphs which are MSO-interpretable in a tree of $\mathcal{T}_n$
- $\mathcal{T}_{n+1} =$ the class of unfoldings of graphs in $\mathcal{G}_n$

Each structure in the pushdown hierarchy has a decidable MSO-theory.

Nontrivial fact:

The sequence $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \ldots$ is strictly increasing.
The First Levels

- $\mathcal{G}_0$ is the class of finite graphs.
- $\mathcal{T}_1$ contains the regular trees.
- $\mathcal{G}_1$ contains the prefix-recognizable graphs.
A Finite Graph, a Regular Tree, a PD Graph
Unfolding Again
Interpretation of Bottom Line

The sequence of leaves defines a copy of the successor structure of the natural numbers.

We give interpretation with regular expressions rather than MSO

Domain expression: \( b + a^*c(d + e)^*f \)

Successor relation:

\[
\overline{bacf} + \\
\overline{fe^*cacf}f + \\
\overline{fe^*dcd}f
\]

Predicate “power of 2”: \( b + a^*cd^*f \)

Result: \((\mathbb{N}, \text{Succ}, \text{Pow}_2)\) is a structure in the Caucal hierarchy.
Towards Factorial Predicate

\((\mathbb{N}, \text{Succ}, \text{Fac})\)

We start as follows:
Obtaining \((\mathbb{N}, +1, \cdot, \text{Fac})\)
Scope of Hierarchy?

The pushdown hierarchy is a very rich class of structures all of which have a decidable MSO-theory.

Open questions:
- Understand which structures belong to the hierarchy
- Compute the smallest level on which a structure occurs

There are structures $S$ which have a decidable monadic theory but do not belong to the hierarchy.

(Example: Consider the set $P$ of iterated 2-powers $1, 2, 2^2, 2^{2^2},$ etc., and take $({\mathbb{N}}, +1, P).$)