Logic, Automata, and Games III: Rabin's Tree Theorem

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Tree Automata

The Model T_2

The structure of the infinite binary tree is

$$\underline{T_2} = (\{0,1\}^*, S_0, S_1, \varepsilon)$$

where S_i is the i-th successor function:

$$S_0(u) = u0, \quad S_1(u) = u1$$

The theory S2S is set of S2S-sentences which are true in $\underline{T_2}$

It is also called the monadic theory of the binary tree,

denoted by $\mathrm{MTh}_2(\underline{T_2})$

Our aim: $\mathrm{MTh}_2(\underline{T_2})$ ist decidable.

End of Büchi's Paper on S1S

Problem 1. Let SC^2 be like SC, except that the functions $2x+1$ and $2x+2$ are taken as primitives in place of $x+1$. Is SC^2 decidable?

This is of some interest, because the functions $2x+1$ and $2x+2$ can be interpreted as the right-successor functions x1 and x2 on the set of all words on two generators 1 and 2.

Problem 2. Let $SC(\alpha)$ be like SC, except that the domain of individuals is the ordinal α, and the well ordering on α is added as a primitive. Is $SC(\omega^2)$ decidable?

As outlined in the introduction, Theorem 2 may be interpreted as a method for deciding whether or not a given finite automaton satisfies a given condition in SC.

Problem 3. Is there a solvability algorithm for SC, i.e., is there a method which applies to any formula $C(\mathbf{i}, \mathbf{u})$ of SC and decides whether or not there is a finite automata recursion $A(\mathbf{i}, \mathbf{r}, \mathbf{u})$ which satisfies the condition C (i.e., $A(\mathbf{i}, \mathbf{r}, \mathbf{u}) \supset C(\mathbf{i}, \mathbf{u})$)?

Example Formulas

Definition of $x \preceq y$ ("node x is prefix of node y"):

$\varphi_s^*(x,y)$ **with** $\varphi_s(z,z') := z0 = z' \vee z1 = z'$

$$\forall X\big((X(y) \wedge \forall z(X(z0) \rightarrow X(z)) \wedge \forall z(X(z1) \rightarrow X(z))) \rightarrow X(x)\big)$$

$\mathrm{Chain}(X)$ **("X is linearly ordered by $\preceq$"):**

$$\forall x \forall y((X(x) \wedge X(y)) \rightarrow (x \preceq y \vee y \preceq x))$$

$\mathrm{Path}(X)$ **("X is a path, i.e. a maximal chain"):**

$$\mathrm{Chain}(X) \wedge \neg \exists Y(X \subseteq Y \wedge X \neq Y \wedge \mathrm{Chain}(Y))$$

$X \subseteq Y$: $\forall z(X(z) \rightarrow Y(z))$

$X = Y$: $\forall z(X(z) \leftrightarrow Y(z))$

Further Formulas

$x < y$ **("x is lexicographically before y"):**

$$\exists z (z0 \preceq x \wedge z1 \preceq y) \vee (x \preceq y \wedge x \neq y)$$

$\mathrm{Finite}(X)$**:**

"each subset Y of X has a minimal and a maximal element w.r.t. $<$"

$$\forall Y((Y \subseteq X \wedge Y \neq \emptyset) \rightarrow$$
$$(\exists y \text{ "}y\text{ is }<\text{-minimal in }Y\text{"} \wedge \exists y \text{ "}y\text{ is }<\text{-maximal in }Y\text{"}))$$

General Format of S2S-Formulas

A formula $\varphi(X_1, \ldots, X_n)$ defines a set of $\{0,1\}^n$-labelled trees.

Such a tree can be presented as a map $T : \{0,1\}^* \to \{0,1\}^n$.

Example:

$\varphi_0(X_1, X_2)$ might express

“X_1 is finite and each path sharing a X_1-element contains infinitely many X_2-elements”

All trees satisfying φ_0 form an S2S-definable tree language.

Rabin introduced tree automata equivalent to S2S.

Format of Tree Automata

$\mathcal{A} = (Q, \Sigma, q_0, \Delta, \mathrm{Acc})$ where

$$\Delta \subseteq Q \times \Sigma \times Q \times Q$$

A transition (q, a, q_1, q_2) allows the automaton in state q at an a-labelled node u to proceed to states q_1, q_2 at the two successor nodes $u0, u1$

A Büchi / Muller / parity tree automaton $\mathcal{A} = (Q, \Sigma, q_0, \Delta, F/\mathcal{F}/c)$ accepts the tree t

if there exists a run ϱ of $\mathcal{A}$ on t such that on each path of ϱ the acceptance condition is satisfied.

Example

$T_1 = \{t \in T^{\omega}_{\{a,b\}} \mid \exists$path through t with infinitely many $b\}$

recognized by a Büchi tree automaton (and thus a parity tree automaton).

"Guess an appropriate path and on it check that infinitely often b occurs by visiting in the next step a final state."

Use states q_a, q_b for the path to guessed, q_0 as initial state, and q for the other nodes.

q_0, q have color 0, q_a color 1, and q_b color 2.

Transitions: $(q_0, a/b, q_{a/b}, q)$, $(q_0, a/b, q, q_{a/b})$,
the same with state q_a and state q_b in place of q_0,
$(q_a, a, q_{a/b}, q)$, $(q_a, a, q, q_{a/b})$, similarly for q_b,
finally $(q, a/b, q, q)$

Example

A parity tree automaton recognizing

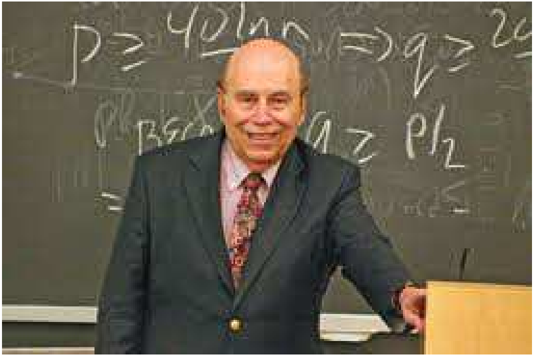
$T_2 = \{t \in T^{\omega}_{\{a,b\}} \mid$
each path through t has only finitely many $b\}$

Use q_a, q_b to signal input letters a, b respectively.

Define $c(q_a) = 0,\ c(q_b) = 1$

The maximal color occurring infinitely often on a path of the run is even (i.e., equal to 0) iff the letter b occurs only finitely often on the path.

Rabin’s Tree Theorem



Michael O. Rabin

Equivalence Logic vs. Automata

(a) A tree language is definable in S2S iff it is recognizable by a parity tree automaton.

(b) The nonemptiness problem for parity tree automata "Given $\mathcal{A}$, does $\mathcal{A}$ accept some tree?" is decidable.

Consequence (from (b) for input-free tree automata):

Rabin's Tree Theorem: MTh(T_2) is decidable.

Everything works as before, but complementation and nonemptiness test are now more difficult.

We use positional determinacy of parity games.

Acceptance via Games

With any parity tree automaton $\mathcal{A} = (Q, \Sigma, q_0, \Delta, c)$

and any input tree t

associate a game $\Gamma_{\mathcal{A},t}$ between two players “Automaton” and “Pathfinder” on the tree t

$\Gamma_{\mathcal{A},t}$

First Automaton picks a transition from Δ which can serve to start a run at the root of the input tree.

Then Pathfinder decides on a direction (left or right) to proceed to a son of the root.

Then Automaton chooses again a transition for this node (compatible with the first transition and the input tree).

Then Pathfinder reacts again by branching left or right from the momentary node, etc.

Play gives a sequence of transitions (and hence a state sequence from Q), built up along a path chosen by Pathfinder.

Automaton wins the play iff the constructed state sequence satisfies the parity condition.

Game Positions

Positions of Automaton are the triples

(tree node w, tree label $t(w)$, state q at w)

By choice of a transition τ of the form $(q, t(w), q', q'')$, a position of Pathfinder is reached.

Positions of Pathfinder are the triples

(tree node w, tree label $t(w)$, transition τ at w)

These positions with the moves define an infinite game graph.

Run Lemma: **The tree automaton $\mathcal{A}$ accepts the input tree t iff in the parity game $\Gamma_{\mathcal{A},t}$ there is a positional winning strategy for player Automaton from the initial position $(\varepsilon, t(\varepsilon), q_0)$**

The Case of Input-Free Automata

We obtain a simpler game $\Gamma_{\mathcal{A}}$, ignoring where we are in the tree.

The game arena is then finite: It consists of

- the set Δ of transitions (q, q', q'')
- the states from Q

The winning condition is the parity condition.

Recall Results on Parity Games

- Parity games are positionally determined: From a given start position one of the two players has a winning strategy, which moreover is positional.

- For parity games over finite game graphs one can decide for any position who wins from this position.

Complementation Proof: Outline

Complementation of tree automata means to express the condition that a given automaton $\mathcal{A}$ does not accept t by acceptance of another automaton.

Non-acceptance by $\mathcal{A}$ means non-existence of a winning strategy for Automaton in $\Gamma_{\mathcal{A},t}$.

Determinacy implies existence of a winning strategy for Pathfinder.

We convert this strategy into an automaton strategy in a different game $\Gamma_{\mathcal{B},t}$.

This gives the desired complement automaton $\mathcal{B}$.

Applying Determinacy (Step 1)

Proof: Let $\mathcal{A} = (Q, \Sigma, q_0, \Delta, c)$ be a parity tree automaton.

We find a parity tree automaton $\mathcal{B}$ accepting precisely the trees $t \in T_\Sigma^\omega$ which are not accepted by $\mathcal{A}$

Start with the following equivalences: For any tree t,

$\mathcal{A}$ does not accept t

iff (by Run Lemma)

Automaton has no winning strategy from the initial position $(\varepsilon, t(\varepsilon), q_0)$ in the parity game $\Gamma_{\mathcal{A},t}$

iff (by Determinacy Theorem)

(*) in $\Gamma_{\mathcal{A},t}$, Pathfinder has a positional winning strategy from $(\varepsilon, t(\varepsilon), q_0)$

Step 2

Reformulate (*) in the form

"$\mathcal{B}$ accepts t" for some tree automaton $\mathcal{B}$

Pathfinder's strategy is a function f from the set $\{0,1\}^* \times \Sigma \times \Delta$ of his vertices into the set $\{0,1\}$ of directions.

Decompose this function into a family

$$(f_w : \Sigma \times \Delta \to \{0,1\})$$

of "local instructions", parameterised by $w \in \{0,1\}^*$

The set I of possible local instructions $i : \Sigma \times \Delta \to \{0,1\}$ is finite,

Thus Pathfinder's winning strategy can be coded by the I-labelled tree s with $s(w) = f_w$

Step 3

Let $s^\wedge t$ be the corresponding $(I \times \Sigma)$-labelled tree with

$$s^\wedge t(w) = (s(w), t(w)) \text{ for } w \in \{0,1\}^*$$

Now (*) is equivalent to the following:

> ***There is an I-labelled tree s such that for all sequences $\tau_0\tau_1\ldots$ of transitions chosen by Automaton and for all (in fact for the unique) $\pi \in \{0,1\}^\omega$ determined by $\tau_0\tau_1\ldots$ via the strategy coded by s, the generated state sequence violates the parity condition.***

This can be checked by a nondeterministic parity tree automaton $\mathcal{B}$, the desired complement automaton for $\mathcal{A}$.

The Input-free Case

An S2S-sentence without free variables will lead to an input-free tree automaton.

An input-free parity tree automaton $\mathcal{A} = (Q, q_0, \Delta, c)$ with $\Delta \subseteq Q \times Q \times Q$ defines the simpler game $\Gamma_{\mathcal{A}}$:

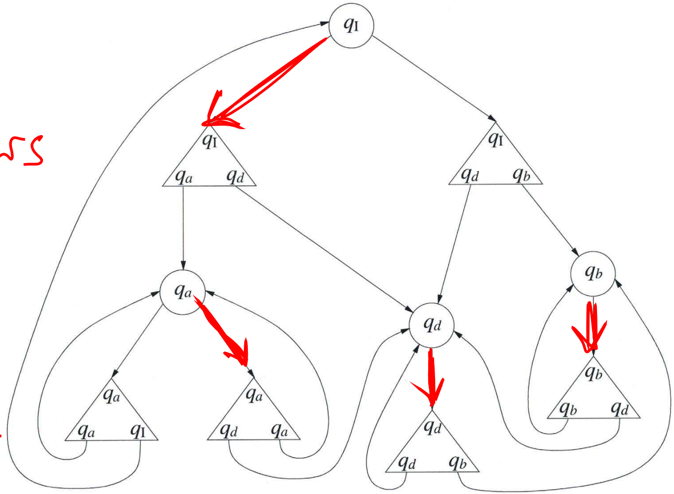
Automaton has positions in in Q and chooses transitions from $Q \times Q \times Q$

Pathfinder has positions in Δ and chooses directions.

Run Lemma (input-free case): $\mathcal{A}$ admits at least one successful run iff Automaton has a winning strategy in $\Gamma_{\mathcal{A}}$ from position q_0.

The first condition is checked effectively.

↓ arrows define strategy of Automaton



Rabin's Tree Theorem

Rabin's Tree Theorem

The theory $\mathrm{MTh}(T_2)$ is decidable.

Proof

Consider an S2S-sentence φ

It can be transformed into an input-free parity tree automaton $\mathcal{A}$ such that

the unlabelled infinite binary tree $\underline{T_2}$ satisfies φ iff $\mathcal{A}$ has some successful run.

The second condition can be checked effectively.

Regular Trees

Rabin's Basis Theorem

Recall: A nonempty regular ω-language contains an ultimately periodic ω-word.

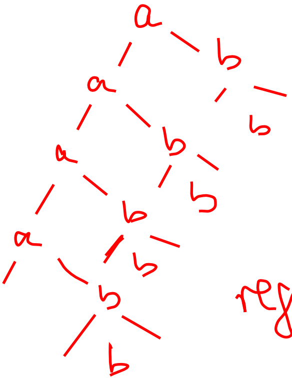
A corresponding result holds for nonempty tree languages which are recognized by parity tree automata.

Rabin's Basis Theorem: A nonempty tree language recognized by a parity tree automaton contains a regular tree.

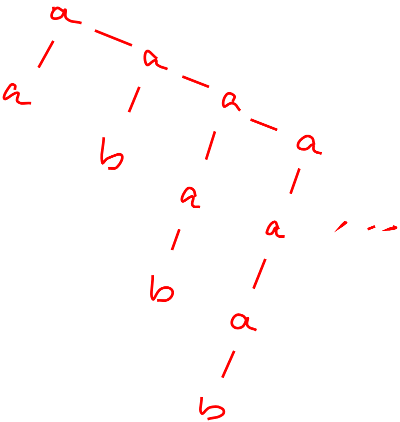
A tree $t \in T_\Sigma^\omega$ is called **regular** if it is "finitely generated" in the following sense:

There is a deterministic finite automaton equipped with output which tells for any given input $w \in \{0,1\}^*$ which label is at node w (i.e. the value $t(w)$).

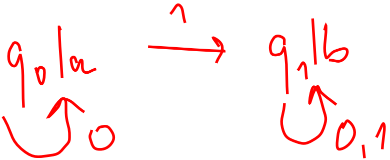
Examples



regular



not regular



Rabin's Basis Theorem: Proof

Assume $\mathcal{A} = (Q, \Sigma, q_0, \Delta, c)$ is a parity tree automaton.

Proceed to an "input-guessing" (and input-free) tree automaton $\mathcal{A}'$ with states in $Q \times \Sigma$:

$\mathcal{A}'$ guesses an input tree and works on it as $\mathcal{A}$ does.

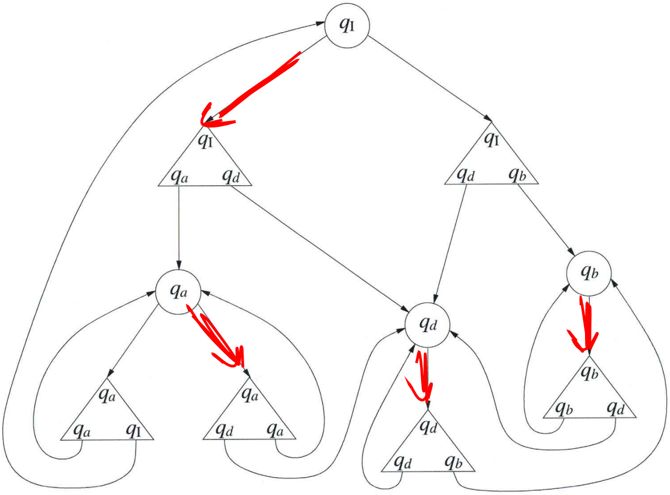
$\mathcal{A}'$ may have several initial states.

Then:

The input-free automaton $\mathcal{A}'$ admits a successful run iff $T(\mathcal{A}) \neq \emptyset$, and a tree in $T(\mathcal{A})$ is extracted from the second components of the run.

Thus a regular tree is generated.

via winning strategy of Automaton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Looking Back

Büchi automata, Muller automata, and parity tree automata provide different versions of quantifier elimination:

to Σ^1_1, to $\mathrm{Bool}(\Pi^0_2)$.

Tree automata provide a less radical way of quantifier elimination than Büchi automata:

An S2S-formula $\varphi(X_1, \ldots, X_n)$ can be transformed into a formula with two second-order quantifiers:

“There is a run on the tree given by $X_1, \ldots, X_n$ such that on each path the acceptance condition is satisfied.”

In logical terminology this is a Σ^1_2-condition.