Logic, Automata, and Games II: Solving Regular Infinite Games

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Preliminaries

Recalling Church's Problem

Given an MSO-formula $\varphi(X, Y)$

we interpret it as the winning condition for Player 2

in the game where Players 1 and 2 build up sequences X, Y bit by bit.

Problem:

Decide whether Player 2 wins

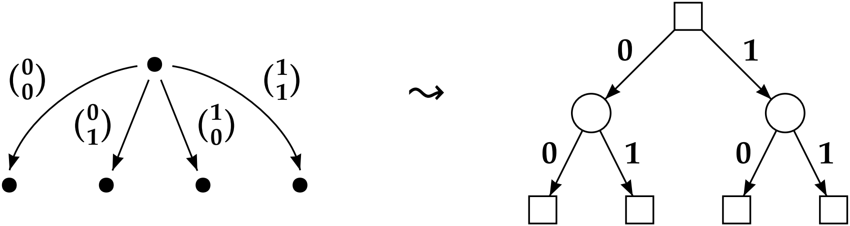
and in this case provide a construction for a finite-state strategy.

Applying McNaughton's Theorem

Transform $\varphi(X,Y)$ into a deterministic Muller automaton $\mathcal{A}$ over the alphabet $\{0,1\}^2$.

$\mathcal{A}$ reads a play and accepts it iff Player 2 is the winner.

Reconfiguration of automaton to "game graph":



A Pioneering Paper

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Project MAC

MAC-M-125

Machine Structures Group Memo No. 14

September, 1965

Finite-State Infinite Games*

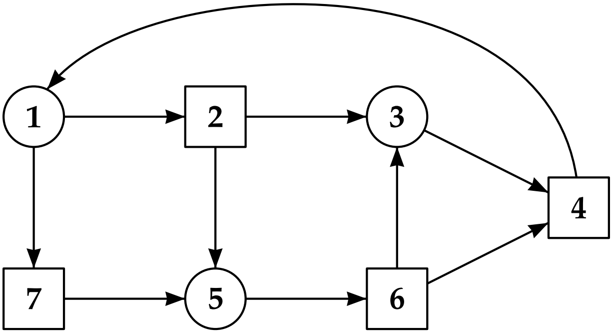
by

Robert McNaughton

This paper will begin with a discussion of infinite games in general, followed by a definition of finite-state infinite games. The main contribution is a proof that every such game has an effectively determinable finite-state winning strategy. The closing remarks establish the Corollary that Büchi's Sequential Calculus has a solvability algorithm, which was an open problem in the Theory of Automata that stimulated interest in the problem of finite-state infinite games.

M.O. Rabin, J. Hartmanis, R. McNaughton

Game Graphs (Arenas)



A game is now given as a pair (G, φ) of a game graph $G = (V, V_1, V_2, E)$ and a winning condition φ for Player 2.

We use only very simple conditions (about infinitely many visits of states during a play).

Muller Condition

For a play $\rho = v_0 v_1 v_2 \ldots$ let $\mathrm{Inf}(\rho) := \{v \in V \mid \exists^\omega i\, v = v_i\}$

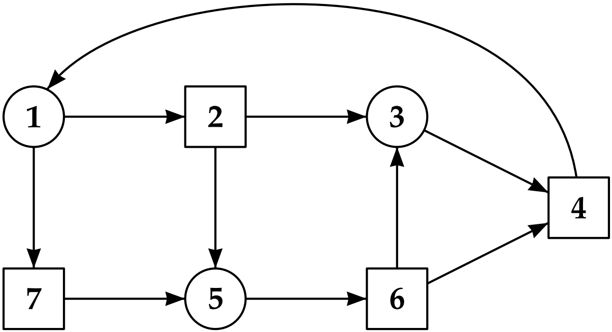
A **Muller condition** is given by a set $\mathcal{F} = \{F_1, \ldots, F_k\}$ with $F_i \subseteq V$ with the interpretation

"Play ρ is won by Player 2 iff $\mathrm{Inf}(\rho)$ is one of the sets F_i"

Solving a game means

- to decide for each vertex v whether Player 2 has a winning strategy for plays starting from v
  ("v belongs to the **winning region** W_2 of Player 2")
- for $v \in W_2$ provide a **winning strategy** for Player 2 from v

Example



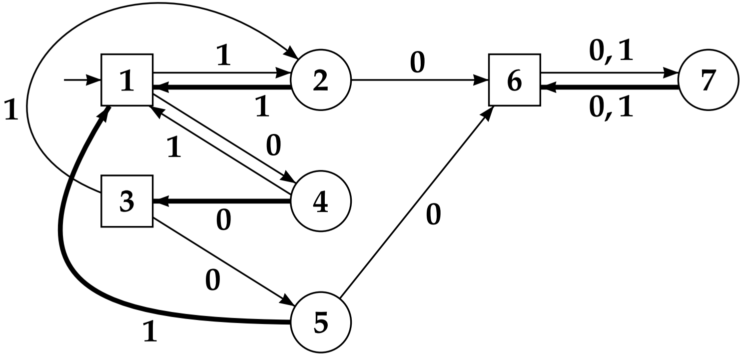
Example 1: “Visit 2 and 6 again and again”

A winning strategy: From 1 go to 2 and 7 in alternation.

Example 2: Visit 2 again and again.

Here Player 2 has a positional (memoryless) winning strategy!

Another Example



$\mathcal{F}$ **contains** $\{1,2,3,4\},\{1,2,3,4,5\},\{1,3,4,5\},\{1,4\}$

Parity Condition

goes back to F. Hausdorff 1914,
introduced as "Rabin chain condition" by A.W. Mostowski 1985
rediscovered as "parity condition" by Emerson and Jutla 1991

We assume a coloring $c : V \to \{1, \ldots, k\}$ of the game graph.

A play $\rho \in V^\omega$ satisfies the parity condition iff the maximal color occurring infinitely often in ρ is even.

The parity condition says for play ρ:

$$\bigvee_{j \text{ even}} (\exists^\omega i : c(\rho(i)) = j \;\wedge \neg \exists^\omega i : c(\rho(i)) > j)$$

A parity game is given by a game graph with finite coloring and the parity condition as winning condition for player 2.

Proof of Büchi-Landweber Theorem

Prelude: Reachability Games

Given a finite game graph $G = (V, V_1, V_2, E)$ and $F \subseteq Q$

Player 2 wins $\rho :\Leftrightarrow \exists i\ \rho(i) \in F$

Inductive construction of $\mathbf{Attr}_2^i(F)$:

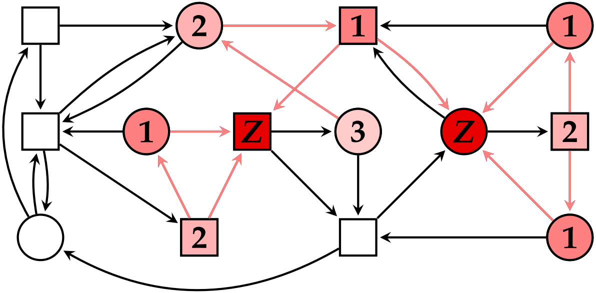
$$
\begin{aligned}
\mathbf{Attr}_2^0(F) &= F, \\
\mathbf{Attr}_2^{i+1}(F) &= \mathbf{Attr}_2^i(F) \\
&\cup \{u \in V_2 \mid \exists (u,v) \in E : v \in \mathbf{Attr}_2^i(F)\} \\
&\cup \{u \in V_1 \mid \forall (u,v) \in E : v \in \mathbf{Attr}_2^i(F)\}
\end{aligned}
$$

$$W_2 = \bigcup \mathbf{Attr}_2^i(F)$$

$$W_1 = V \setminus \mathbf{Attr}_2(F)$$

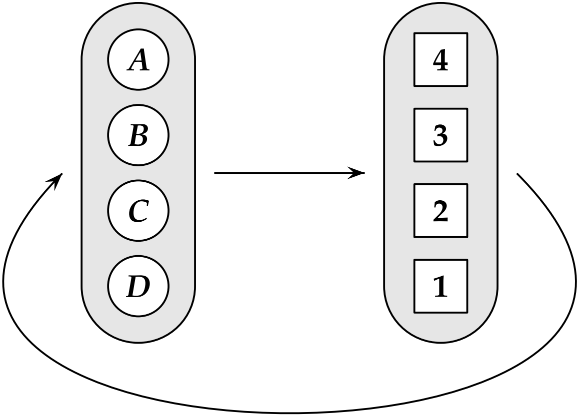
Over infinite graphs, a corresponding transfinite induction works.

Example



Towards Parity Games: DJW-Game

invented by Dziembowski, Jurdzinski and Walukiewicz (1997)



Winning condition:

$$|\mathrm{Inf}(\rho) \cap \{A, B, C, D\}| = \max(\mathrm{Inf}(\rho) \cap \{1, 2, 3, 4\})$$

Latest Appearance Record

	D	B	B	D	C	B	A	B	A	C	B	A	B	A	C
A	D	B	$\underline{B}$	D	C	B	A	B	A	C	B	A	B	A	C
B	A	D	D	$\underline{B}$	D	C	B	$\underline{A}$	$\underline{B}$	A	C	B	$\underline{A}$	$\underline{B}$	A
C	B	$\underline{A}$	A	A	B	$\underline{D}$	C	C	C	$\underline{B}$	$\underline{A}$	$\underline{C}$	C	C	$\underline{B}$
$\underline{D}$	$\underline{C}$	C	C	C	$\underline{A}$	A	$\underline{D}$	D	D	D	D	D	D	D	D

Solution of the DJW-Game

LAR-strategy for Player 2:

During play, update and use the LAR as follows:

- **shift the current letter vertex to the front**
  underline the position from where the current letter was taken
- **move to the number vertex given by underlined position**

These are the two items performed by the strategy:

- **update of memory**
- **choice of next step ("output")**

Result: Finite-state winning strategy with $n! \cdot n$ states for a game graph with $2n$ vertices

Analyzing the Winning Strategy

Call the underlined position the hit

The states of a LAR up to the hit are called the recent states.

The Muller winning condition says:

For the highest hit occurring infinitely often, the corresponding recent states form a set in $\mathcal{F}$.

We merge the hit value h and the status of the recent states into a LAR-color:

- **color $2h$ if the recent states form set in $\mathcal{F}$**
- **color $2h-1$ otherwise**

So the Muller winning condition says:

The highest LAR-color occurring infinitely often is even

In General: Memory Extensions

Use (finite) memory space S, initialized with s_0

Transform game graph $G = (V, V_1, V_2, E)$ into
$G' = (S \times V, S \times V_1, S \times V_2, E')$

with memory update function $\delta : S \times V \rightarrow S$

For $(u, v) \in E$ put $((s, u), (\delta(s, u), v))$ into E'

Each play ρ over G induces a play ρ' over G'.

Write $(G, \varphi) \leq (G', \varphi')$ if for each ρ:

Player 2 wins ρ w.r.t. φ iff Player 2 wins ρ' w.r.t. φ'

Application of Game Reduction

Assume $(G, \varphi) \leq (G', \varphi')$ via memory extension with finite S.

If (G', φ') is determined with memoryless winning strategies, then (G, φ) is determined with finite-state strategies.

Proof

Use state set S of memory expansion.

Use memory update for transition function.

Use memoryless winning strategy for definition of output function $\eta : S \times V_2 \to V_1$

If $(s, u) \to (s', v)$ is an edge of the winning strategy then define $\eta(s, u) = v$.

Positional Determinacy of Parity Games

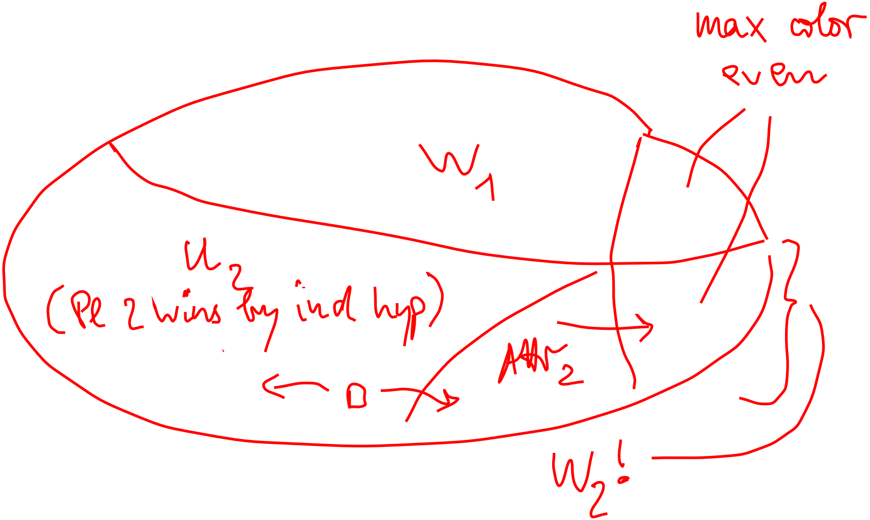
Theorem (Emerson-Jutla 1991)

- **Parity games are determined (i.e., each vertex belongs to W_1 or W_2), and the winner from a given vertex has a positional winning strategy.**
- **Over finite graphs, the winning regions and winning strategies of the two players can be computed in (at most) exponential time in the number of vertices of the game graph.**

Whether polynomial time suffices in the second item, is open.

Positional Determinacy of Parity Games

by induction on number of colors, say with even highest color



Finite Arenas: Find Winning Regions

1. **Guess W_1 and W_2 and positional strategies given by edge sets: E_1 and E_2**
2. **Check that E_1 defines a winning strategy from each $q \in W_1$ and that E_2 a winning strategy from each $q \in W_2$**

Pursue Step 2: Check whether a given positional strategy is a winning strategy for Player 2 from q.

Checking Strategies

Remark: For a fixed positional strategy f of Player 2 one can decide in polynomial time for any $q \in Q$, whether f is a winning strategy from q

Proof:

Consider the graph G_2 induced by a given by a positional strategy of Player 2. We have a one-player game (of Player 1).

Check whether in G_2 there is a path from q to a loop whose highest color is odd.

For this, check for reachable SCC's in the restriction of G_2 to colors $\leq m$ for odd m.

Decision Problem "Parity Game"

Given: A finite game graph G with coloring, $q \in Q$

Question: Does player 2 win the parity game from q?
(Short: "$q \in W_2$ in the corresponding parity game?")

Theorem: The Problem "Parity Game" is in the complexity class NP $\cap$ co-NP

Proof: The above nondeterministic procedure shows that the problem is in NP.

It remains to show that the complementary problem "$q \notin W_2$?" is also in NP.

This problem means "$q \in W_1$?". It is solvable in the same way as "$q \in W_2$?", hence in NP.

Intriguing: Open problem: Is "parity game" in P?