The problem of absolute generality - 4

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Review

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- **Generality relativism** has been rejected: inexpressibility problem
- **Orthodox generality absolutism** has been rejected: unstable vis-à-vis paradox of indefinite extensibility; inexpressibility problem

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For instance,

$$\forall X(D(X) \leftrightarrow \exists x \forall y(y \in x \leftrightarrow Xy)) \tag{1}$$

$$\forall X(X \subseteq On \rightarrow (D(X) \leftrightarrow \exists x \forall y(y = LUB(X))) \tag{2}$$

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$$\neg\Diamond\forall m\exists n\, \textsc{Successor}(m, n) \tag{4}$$

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Our analysis will have to be integrated with the non-modal language of standard mathematics.

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1. Mathematical objects are introduced successively through a well-ordered series of stages.
2. The introduction of a mathematical object consists in the specification of a (permanent) identity condition.
3. *Cumulativity.* The licence to introduce an object never goes away but can always be exercised at a later stage.
4. *Maximality* (optional). At every stage we introduce all the mathematical objects we are entitled to introduce.

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- An **accessibility relation** $w \leq w'$) which holds iff w' is a (not necessarily proper) extension of w. So $\leq$ is
  - reflexive, anti-symmetric, and transitive.
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  - reflexive, anti-symmetric, and transitive.
  - directed (i.e. $\exists z(x \leq z \wedge y \leq z)$)
  - well-founded.
- The resulting **Kripke-models** validate the modal logic S4.2 = S4 + (G):

$$\Diamond\Box p \rightarrow \Box\Diamond p. \tag{G}$$

Actualist and potentialist theories compared (I)

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- The *modal translation* $\phi \mapsto \phi^{\Diamond}$ replaces each ordinary quantifier with the corresponding modalized quantifier.
- A formula is *fully modalized* iff all of its quantifiers are modalized.
- A formula $\phi(\mathbf{u})$ is *stable* iff the following two conditions hold:

$$\phi(\mathbf{u}) \to \Box\phi(\mathbf{u}) \qquad (\mathrm{S}\textsc{tb}^{+}\text{-}\phi)$$

$$\neg\phi(\mathbf{u}) \to \Box\neg\phi(\mathbf{u}) \qquad (\mathrm{S}\textsc{tb}^{-}\text{-}\phi)$$

Actualist and potentialist theories compared (II)

Lemma

Let ϕ be a fully modalized formula of a modal language $\mathcal{L}^{\Diamond}$. Then S4.2 and the stability axioms for $\mathcal{L}^{\Diamond}$ prove

- *that $\Diamond\phi$, ϕ, and $\Box\phi$ are equivalent,*
- *that ϕ is stable.*

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Theorem (Mirroring)

Let $\vdash^{\Diamond}$ be provability by $\vdash$, S4.2, and axioms stating that every atomic predicate is stable, but with no higher-order comprehension. Then we have:

$$\phi_1, \ldots, \phi_n \vdash \psi \quad \textit{iff} \quad \phi_1^{\Diamond}, \ldots, \phi_n^{\Diamond} \vdash^{\Diamond} \psi^{\Diamond}.$$

Plural logic (I)

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- Let $\mathcal{L}_{PFO}$ be the language which adds to $\mathcal{L}_{FO}$ plural variables and quantifiers binding these variables and the logical predicate $\prec$.
- Let $\mathcal{L}^{\Diamond}_{PFO}$ be the modal language which results from adding to $\mathcal{L}_{PFO}$ the two modal operators.

Plural logic (II)

Definition (The theories of plural logic)

Let PFO be the $\mathcal{L}_{PFO}$-theory consisting of first-order logic, the standard rules for the plural quantifiers, and the comprehension scheme:

$$\exists xx \forall u[u \prec xx \leftrightarrow \phi(u)] \qquad \text{(P-Comp)}$$

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Let MPFO be the $\mathcal{L}^{\Diamond}_{PFO}$-theory which adds to PFO the modal logic S4.2 and the following axioms:

$$u \prec xx \rightarrow \Box(u \prec xx) \tag{$\textsc{Stb}^+ \prec$}$$

$$u \not\prec xx \rightarrow \Box(u \not\prec xx) \tag{$\textsc{Stb}^- \prec$}$$

$$\forall u (u \prec xx \rightarrow \Box\theta) \rightarrow \Box \forall u (u \prec xx \rightarrow \theta) \tag{$\textsc{Cl}$-$\prec$}$$

A modal characterization of 'definiteness' (I)

Assuming that pluralities are rigid with respect to the modality, we can explicate the idea that there is a definite 'collection' of ϕs as follows:

$$Def_x(\phi) :\leftrightarrow \Diamond\exists xx\Box\forall y(y \prec xx \leftrightarrow \phi^{\Diamond}[y/x])$$

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This analysis implies various useful principles of definiteness (aka. plural comprehension), e.g.:

$$Def_x(x \neq x) \quad \text{(D-Empty)}$$

$$Def_x\phi \rightarrow \forall a\, Def_x(\phi \vee x = a) \quad \text{(D-Adj)}$$

$$Def_x\phi \rightarrow Def_x(\phi \wedge \psi) \quad \text{(D-Sep)}$$

$$Def_x\phi \wedge Def_x\psi \rightarrow Def_x(\phi \vee \psi) \quad \text{(D-Union)}$$

A modal characterization of 'definiteness' (II)

It is natural to add a Replacement principle as well:

$$Def_x(\phi(x)) \wedge \forall x(\phi(x) \rightarrow Def_y \psi(x, y)) \rightarrow Def_y \exists x(\phi(x) \wedge \psi(x, y))$$

Applications of this notion of definiteness

- a stable and well-motivated way to restrict the extensibility principles
- modal set theory (Linnebo, 2013)
- a property theory

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A stable way to restrict the extensibility principles

- X is definite iff it is possible for X to be completed as a plurality

$$D(X) :\leftrightarrow \Diamond \exists xx \Box \forall y (y \prec xx \leftrightarrow Xy)$$

- As argued in lecture 2, a plurality $\alpha\alpha$ of ordinal numbers can be given a LUB, and *mutatis mutandis* for sets
- But the arguments showing this work only for pluralities, not for indefinite X

The nature of sets

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Then the extensionality and extensional definiteness of sets can be given the following alternative formulations:

$$x \equiv uu \wedge y \equiv vv \to (x = y \leftrightarrow uu \approx vv) \qquad \text{(Ext)}$$

$$x \equiv uu \to \Box(x \equiv uu) \qquad (\textsc{Stb}^{+}\text{-}\equiv)$$

$$x \not\equiv uu \to \Box(x \not\equiv uu) \qquad (\textsc{Stb}^{-}\text{-}\equiv)$$

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Assume membership is now defined by:

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Priority of elements to set

$$\forall x[\exists y(y \in x) \rightarrow \exists y(y \in x \wedge \forall z(z \in x \rightarrow z \notin y))] \tag{F}$$

Subsethood

Consider the theory resulting from the axioms just laid out. This theory proves that subsethood is stable

$$u \subseteq a \rightarrow \Box(u \subseteq a) \quad (\text{Stb}^{+}\text{-}\subseteq)$$

$$u \not\subseteq a \rightarrow \Box(u \not\subseteq a) \quad (\text{Stb}^{-}\text{-}\subseteq)$$

but not that it is closed

$$\forall u(u \subseteq a \rightarrow \Box\theta) \rightarrow \Box\forall u(u \subseteq a \rightarrow \theta) \quad (\text{Cl-}\subseteq)$$

The principle of set existence

Any available objects *may* form a set:

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Let MST$^-$ be the resulting modal set theory. Let MST be the result of adding an axiom to the effect that $Def_x(x \subseteq a)$ for any a.

Theorem

Assume MST$^-$ proves that the condition $\phi(u)$ is definite. Then this theory also proves the following:

$$\Diamond\exists y\Box\forall u(u \in y \leftrightarrow \phi(u)) \tag{6}$$

Zermelo set theory - Infinity

Theorem

MST proves the modal translations of all the axioms of Zermelo set theory minus Infinity; that is, Extensionality, Foundation, and the following set existence claims:

$$\exists x \forall u\,(u \notin x) \qquad \text{(Empty Set)}$$

$$\exists x \forall u\,(u \in x \leftrightarrow u = a \vee u = b) \qquad \text{(Pairs)}$$

$$\exists x \forall u\,(u \in x \leftrightarrow \exists v(u \in v \wedge v \in a)) \qquad \text{(Union)}$$

$$\exists x \forall u\,(u \in x \leftrightarrow u \in a \wedge \phi(u)) \qquad \text{(Separation)}$$

$$\exists x \forall u\,(u \in x \leftrightarrow u \subseteq a) \qquad \text{(Power)}$$

MST^{-} proves the same except (the translation of) (Power).

Zermelo-Fraenkel set theory

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Theorem

(a) *Adding ($\Diamond$-Refl$_0$) allows us to prove Infinity$^{\Diamond}$.*

(b) *Adding ($\Diamond$-Refl) allows us to prove Replacement$^{\Diamond}$.*

The reverse interpretation

Theorem

The potentialist theory MST + ($\Diamond$-Refl) is interpretable in the actualist theory ZF and is therefore consistent provided ZF itself is.

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Objects that can represent indefinite 'collections', such as the domain of absolutely everything and the intended interpretation of $\in$

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An outline of a theory of properties

- '$x \ \eta \ y$' for 'x instantiates y' (cf. '$x \in y$' in set theory)
- We generate ever larger interpretations E of η
- A condition ϕ is *grounded relative to E* iff for any permissible extension F of E, ϕ has the same extension under E as under F, i.e.

$$(\forall F \sqsupseteq E)([\![\phi]\!]_E = [\![\phi]\!]_F)$$

Properties (II)

Some theories of properties: (Fine, 2005), (Linnebo, 2006), (Linnebo, 2009)

$$\exists x \forall y (y \ \eta \ x \leftrightarrow \phi(y)) \qquad \text{only where } \phi \text{ is grounded}$$

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We get a sequence T_α of stronger and stronger theories of properties, such that each $T_{\alpha+1}$ proves the existence of an interpretation on which T_α is true.

Taking stock

What has been done/argued

- Absolute generality remains a huge problem: the new orthodoxy is not much better off than generality relativism
- The only solution seems to be a type-free account of absolute generality.
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Where more work is needed

- Better theories of properties.
- An account of the modality
- 'Revenge problems', esp. what about the interpretation of my 'η'?

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