The problem of absolute generality - 3

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4. The only objects that have members are sets, as understood on the iterative conception.

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Proof. By claim 1, we can quantify over all sets. By claim 2, there is a domain whose members are all and only the sets. By claim 3, this domain is an object. And by claim 4, this object has to be a set S. Since S must have been in the range of the quantification with which we started, it is a member of the domain and thus also an element of itself. But self-membership is impermissible for sets as understood on the iterative conception, which by claim 4 includes our set S. $\dashv$

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- In a similar manner, we deny the Extensibility assumption of the paradoxes of indefinite extensibility (although this was criticized in the previous lecture).
- This involves a very substantive use of plural and higher-order logic.
- The response to the paradox of absolute generality will be criticized below.

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- This is my preferred view (Linnebo, 2006) and will be developed in lecture 4.

Problems with generality relativism

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- Consider the claim that my current language does not quantify over everything.
- This entails that there is something over which my quantifiers do not range.
- But here I use a quantifier to assert the existence of something not in the range of my quantifiers!

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- But in so expressing the view, one is violating the view: interpretations are as indefinitely extensible as (ordinal) numbers and sets.
- If '$\forall I$' doesn't range over absolutely all interpretations, then (1) is compatible with relativism.
- But now (1) fails to express the intended view. All that is expressed is that every interpretation *in some limited range of interpretations* can be expanded.

Schematic generality

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Generality relativism can now be expressed as follows:

$$I \subset I^+ \tag{2}$$

where $I \mapsto I^+$ is a definable operation on interpretations.

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Some definitions: Truth theory

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A *truth theory* for a language $\mathcal{L}$ in another language $\mathcal{L}'$ consists of an $\mathcal{L}'$-theory which proves all instances of the following:

$$\ulcorner \phi \urcorner \text{ is true } \leftrightarrow \mathit{tr}(\phi) \tag{3}$$

where 'x is true' is an open formula of $\mathcal{L}'$, and *tr* is a translation from $\mathcal{L}$ to $\mathcal{L}'$.

Some definitions: Generalized semantics

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A generalized semantics does in cases of absolute generality what model theory does in cases of set-sized domains.

A closer look at the type theoretic response

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We have already discussed Premise 1. But let's have a closer look at Premise 2 and the Ascent Theorem.

Kreisel's squeezing argument

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Finally, by soundness we have:

$$\Gamma \vdash \phi \text{ implies } \Gamma \Rightarrow \phi,$$

which closes the circle of implications.

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But we get the same effect by means of **reflection principles**, which assert that any sentence that is true in the universe of sets is also true in some set-based model:

$$\phi \rightarrow \exists\alpha\,(\phi^{V_\alpha}) \tag{4}$$

where ϕ^{V_α} is the result of restricting the quantifiers of ϕ to V_α. (See (Shapiro, 1987) and (Burgess, 2004).)

Why we should still insist on a generalized semantics

We would like a theory of logical consequence which is not only extensionally but also intensionally correct.

- By the downwards Löwenheim-Skolem theorem, the definition of logical consequence could make do with countable structures.
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Generalized semantics is legitimate and interesting in its own right.

- Our language has one interpretation. But there are myriad others it might have had.
- It is a legitimate mathematical undertaking to study all these interpretations and examine how the truth of sentences is affected by the choice of interpretation.

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It is now straightforward to express that some ordered pairs *mm* code a surjective map of members of a plurality to its subpluralities.

Plural Cantor (II)

Theorem (Plural Cantor's theorem)

Let xx be a plurality with at least two members, and assume the domain has a pairing function. Then xx has more subpluralities than it has members in the precise sense that there is no plurality that codes a surjective mapping from xx to subpluralities of xx.

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Proof. Let $x \equiv xx$ abbreviate (5), and let $\delta(x)$ be the claim that x is not among any plurality that it codes, that is:

$$\forall xx(x \equiv xx \rightarrow x \not\prec xx).$$

I claim that $\delta(x)$ is satisfied by at least one object. Consider two distinct objects a and b. The claim follows by observing that there are three pluralities all of whose members are among a and b, which are just two objects. Having established the claim, we can use plural comprehension to let dd be the 'diagonal plurality' comprising all and only the objects that satisfy $\delta(x)$. Then we get $d \prec dd \leftrightarrow d \not\prec dd$. ⊣

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Proof. Under the assumption of absolute generality, an ordinary singular predicate can be interpreted by means of any plurality. But by the generalization of Cantor's theorem, there are more pluralities than there are objects. It follows that interpretations of a first-order language cannot be objects but must be represented by means of higher-order resources. ⊣

Some generalized ascent theorems

Theorem (Generalized Ascent Theorem)

(a) *A truth theory for an n-th order language $\mathcal{L}$ can be developed in the language that results from $\mathcal{L}$ by either adding an n-th order predicate or adding variables and quantifiers of order $n + 1$.*

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(c) *A model theory for a general n-th order language $\mathcal{L}$ can be developed in a language of order $n+1$ with no non-logical constants.*

(d) *These results are optimal.*

The origins of these results

(Boolos, 1985) is a (re-)discovery of (a) for $n = 2$ on the plural interpretation of orders. (Rayo and Uzquiano, 1999) is a (re-)discovery of (b) and (c) for $n = 2$. (Rayo, 2006) proves the theorem in the generality stated above.

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The proofs are somewhat involved but can be broken down into components.

- (a) = (Tarski, 1935) + (Frege, 1879)/(Dedekind, 1888) + coding
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In fact, (Linnebo and Rayo, 2012) provide generalizations into the transfinite.

Theorem (Transfinite Ascent Theorem)

(a) *We cannot develop a generalized semantics for a language of order α in another language of order α.*

(b) *For any successor ordinal α, we can develop a generalized semantics for a language of order α in a language of order $\alpha + 1$.*

Component 1: Tarski on satisfaction and model theory (I)

- We need a variable assignment A, which assigns an entity of appropriate sort to each variable.
- When we do model theory, we also consider a model M that specifies a domain and an assignment of appropriate entities to all constants of the language.
- Assume we can define the notions of an assignment and a model, as well as the assignment $[\![E]\!] = [\![E]\!]_{A,M}$ made to an expression E relative to the variable assignment A and model M.
- Assume the n-tuple of any entities can be coded by means of an entity whose order is no higher than that of the given entities.

Component 1: Tarski on satisfaction and model theory (I)

Definition

We define the notion of satisfaction in a model M relative to an assignment A as follows:

1. *If ϕ is a formula of the form $t(t_1, \ldots, t_n)$ for an n-place term t and where the t_i are of appropriate order, then:*
   $Sat\langle \ulcorner \phi \urcorner, A\rangle$ *iff* $\langle [\![t_1]\!], \ldots, [\![t_n]\!]\rangle \; \eta \; [\![t]\!]$
2. *If ϕ is a formula of the form $t_1 = t_2$ for two terms of the same order, then:*
   $Sat\langle \ulcorner \phi \urcorner, A\rangle$ *iff* $[\![t_1]\!] = [\![t_2]\!]$
3. *If ϕ is a formula of the form $\neg\psi$, then:*
   $Sat\langle \ulcorner \phi \urcorner, A\rangle$ *iff* $\neg Sat\langle \ulcorner \psi \urcorner, A\rangle$
4. *If ϕ is a formula of the form $\psi_1 \wedge \psi_2$, then:*
   $Sat\langle \ulcorner \phi \urcorner, A\rangle$ *iff* $Sat\langle \ulcorner \psi_1 \urcorner, A\rangle$ *and* $Sat\langle \ulcorner \psi_1 \urcorner, A\rangle$
5. *If ϕ is a formula of the form $\exists v\, \psi$ for some variable v, then:*
   $Sat\langle \ulcorner \phi \urcorner, A\rangle$ *iff there is a v-variant B of A, which assigns to v*

Component 2: Frege & Dedekind on recursive def's (I)

(Frege, 1879) and (Dedekind, 1888) discovered that recursive definitions can be turned into explicit ones by generalization over "collections" of the entities related by the recursive definition.

Example

Let $\textsc{Add}(x, y, z)$ *("z is a sum of x and y") be defined as follows:*

- $\textsc{Add}(x, 0, x)$
- $\textsc{Add}(x, y, z) \rightarrow \textsc{Add}(x, y', z')$

Then $\textsc{Add}(x, y, z)$ *can be defined explicitly as follows:*

$$\forall R[\forall u R(u, 0, u) \wedge \forall u, v, w(R(u, v, w) \rightarrow R(u, v', w')) \rightarrow R(x, y, z)]$$

Component 2: Frege & Dedekind on recursive def's (II)

(Tarski, 1935) realized that his own recursive definition of satisfaction could be turned into an explicit one in this way.

- Replace $Sat(\ulcorner\phi\urcorner, A)$ with $Y\langle\ulcorner\phi\urcorner, A\rangle$ in each clause of the recursive definition (1).
- Conjoin the resulting clauses and abbreviate the resulting conjunction $\Phi(\ulcorner\phi\urcorner, A)$.
- Then $Sat(\ulcorner\phi\urcorner, A)$ can be defined explicitly as:

$$Sat(\ulcorner\phi\urcorner, A) \;:\leftrightarrow\; \forall Y[\Phi(\ulcorner\phi\urcorner, A) \to Y\langle\ulcorner\phi\urcorner, A\rangle] \tag{6}$$

Component 3: Some coding (I)

Example

(Boolos, 1985) realizes that an assignment to plural variables can be taken to consist of some (set-theoretic) ordered pairs aa of the form $\langle v, x \rangle$ where v is a plural variable, subject to the requirement that for every plural variable v, there is at least one pair of the form $\langle v, x \rangle$ among aa.

We can even talk about the assignment $[\![v]\!]_{aa}$ made to the plural variable v by aa:

$$\forall x[x \prec [\![v]\!]_{aa} \leftrightarrow \langle v, x \rangle \prec aa] \tag{7}$$

Likewise, it is straightforward to define the notion of a model.

Component 3: Some coding (II)

Theorem (*n*-tuples)

Assume that for any two objects there is another object that serves as their ordered pair. Given any entities $x^{k_1}, \ldots, x^{k_n}$, we can then code for the ordered n-tuple of these entities by means of a single entity x^k, where k is the maximum of the k_i. We will designate this entity x^k as $\langle x^{k_1}, \ldots, x^{k_n} \rangle$.

Proof idea. The key step is to define the ordered pair of an object a and an n-th order entity x^n. We proceed by induction. Assume we can have defined the ordered pair of a with any n-th order entity. Then we let the ordered pair $\langle a, x^{n+1} \rangle$ of a with an entity x^{n+1} of order $n+1$ be the unique entity y^{n+1} such that:

$$\forall u^n [y^{n+1}(u^n) \leftrightarrow \exists x^n (u^n = \langle a, x^n \rangle \wedge x^{n+1}(x^n))] \tag{8}$$

Proving the main theorem

Proof sketch for Theorem 2 (b) and (c). Insert upper indices for orders in Definition 1 and use the three components to keep track of how high orders are needed to formulate the recursive definitions in questions and turn them into explicit ones. (For a detailed spelling out of this strategy, see (Linnebo and Rayo, 2012).)

Review of the situation

- We want a generalized semantics for our first-order language.

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- Cannot properly state the view. (Cf. someone who regards the higher-order quantifiers as just restricted first-order quantifiers.)

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