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3. The domain is an object, in the sense that it can be the value of a singular first-order variable.

4. The only objects that have members are sets, as understood on the iterative conception.
Proposition

The four claims just listed are inconsistent.

Proof. By claim 1, we can quantify over all sets. By claim 2, there is a domain whose members are all and only the sets. By claim 3, this domain is an object. And by claim 4, this object has to be a set $S$. Since $S$ must have been in the range of the quantification with which we started, it is a member of the domain and thus also an element of itself. But self-membership is impermissible for sets as understood on the iterative conception, which by claim 4 includes our set $S$. $\Box$
The paradox of absolute generality (II)

**Proposition**

*The four claims just listed are inconsistent.*

**Proof.** By claim 1, we can quantify over all sets. By claim 2, there is a domain whose members are all and only the sets. By claim 3, this domain is an object. And by claim 4, this object has to be a set $S$. Since $S$ must have been in the range of the quantification with which we started, it is a member of the domain and thus also an element of itself. But self-membership is impermissible for sets as understood on the iterative conception, which by claim 4 includes our set $S$. $\neg$
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- It provides no help with the paradoxes of indefinite extensibility.
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Responses to our paradoxes (II)

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- This involves a very substantive use of plural and higher-order logic.
- The response to the paradox of absolute generality will be criticized below.
Intensionalists: deny claim 4 (only sets have members).

- Distinguish between extensional sets and intensional properties.
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- This is my preferred view (Linnebo, 2006) and will be developed in lecture 4.
Problems with generality relativism

Absolute generality appears to be possible and needed. (Lecture 1)
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“Generality relativism is that the view cannot be coherently stated”.

Maybe the singularist replies that some mystical censor stops us from quantifying over absolutely everything without restriction. Lo, he violates his own stricture in the very act of proclaiming it! (Lewis, 1991, p. 68)

Consider the claim that my current language does not quantify over everything. This entails that there is something over which my quantifiers do not range. But here I use a quantifier to assert the existence of something not in the range of my quantifiers!
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Quantification over interpretations to the rescue?

Let $I \subset J$ abbreviate $\exists J x \forall I y (x \neq y)$.
Let \( I \subset J \) abbreviate \( \exists J \times \forall I y(x \neq y) \). One may then attempt to express generality relativism as follows:

\[
\forall I \exists J (I \subset J)
\]  (1)

If '\( \forall I \)' ranges over absolutely all interpretations, then (1) expresses what it is meant to express. But in so expressing the view, one is violating the view: interpretations are as indefinitely extensible as (ordinal) numbers and sets. If '\( \forall I \)' doesn't range over absolutely all interpretations, then (1) is compatible with relativism. But now (1) fails to express the intended view. All that is expressed is that every interpretation in some limited range of interpretations can be expanded.
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Generality relativism can now be expressed as follows:

\[ I \subset I^+ \]  

(2)

where \( I \mapsto I^+ \) is a definable operation on interpretations.
Problems with schematic generality

Schematic generality enables us to express absolutely general $\Pi_1$-sentences, but not $\Sigma_1$ or beyond.
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- Schematic generality enables us to express absolutely general \( \Pi_1 \)-sentences, but not \( \Sigma_1 \) or beyond.

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Problems with schematic generality

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- Any truth-evaluable claim *can* be negated.

Before looking at the type theoretic response, we need some definitions.
Some definitions: Truth theory

A truth theory for a language $L$ in another language $L'$ consists of an $L'$-theory which proves all instances of the following:

\[ \text{⌜φ⌝ is true} \leftrightarrow \text{tr(φ)} \] (3)

where 'x is true' is an open formula of $L'$, and $\text{tr}$ is a translation from $L$ to $L'$. 

Øystein Linnebo (Oslo and London)
A *truth theory* for a language $\mathcal{L}$ in another language $\mathcal{L}'$ consists of an $\mathcal{L}'$-theory which proves all instances of the following:

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where ‘$x$ is true’ is an open formula of $\mathcal{L}'$, and $tr$ is a translation from $\mathcal{L}$ to $\mathcal{L}'$. 

Some definitions: Truth theory
Let a *generalized semantics* be a theory of all possible interpretations that a language might take, without any artificial restrictions on the domains, interpretations, and variable assignments.
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A generalized semantics does in cases of absolute generality what model theory does in cases of set-sized domains.
A closer look at the type theoretic response

Type theorists claim that a proper semantics for a first-order language requires higher-order resources.

Premise 1
Absolute generality is possible.

Premise 2 (Semantic optimism)
For any legitimate language, we need a generalized semantics.

Ascent Theorem (basic form)
If absolute generality is possible, then a generalized semantics for a first-order language cannot be given in another first-order language but can be given in a higher-order language.

We have already discussed Premise 1. But let’s have a closer look at Premise 2 and the Ascent Theorem.
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Kreisel’s squeezing argument

The argument shows that model theory (as opposed to a generalized semantics) suffices for the definition of logical consequence for first-order languages.

Let $\Gamma \Rightarrow \phi$ formalize the claim that in any structure in which each member of $\Gamma$ is true, whether set-based or not, $\phi$ is also true. Since every set-based structure is a structure, we have: $\Gamma \Rightarrow \phi$ implies $\Gamma \models \phi$.

By the completeness theorem for FOL, we have: $\Gamma \models \phi$ implies $\Gamma \vdash \phi$.

Finally, by soundness we have: $\Gamma \vdash \phi$ implies $\Gamma \Rightarrow \phi$, which closes the circle of implications.
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For higher-order languages (with standard semantics), Kreisel’s argument doesn’t work, as there is no completeness theorem.
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But we get the same effect by means of **reflection principles**, which assert that any sentence that is true in the universe of sets is also true in some set-based model:

\[ \phi \rightarrow \exists \alpha (\phi^V_\alpha) \]  

(4)

where \( \phi^V_\alpha \) is the result of restricting the quantifiers of \( \phi \) to \( V_\alpha \). (See (Shapiro, 1987) and (Burgess, 2004).)
Why we should still insist on a generalized semantics

We would like a theory of logical consequence which is not only extensionally but also intensionally correct.

- By the downwards Löwenheim-Skolem theorem, the definition of logical consequence could make do with countable structures.
- Although extensionally correct, this definition fails to capture the intended intension.

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Generalized semantics is legitimate and interesting in its own right.

- Our language has one interpretation. But there are myriad others it might have had.
- It is a legitimate mathematical undertaking to study all these interpretations and examine how the truth of sentences is affected by the choice of interpretation.
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Say that some ordered pairs \( mm \) code a map that sends \( x \) to \( xx \) iff:

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Say that some ordered pairs $mm$ code a map that sends $x$ to $xx$ iff:

$$\forall y (y < xx \iff \langle x, y \rangle < mm).$$ (5)

It is now straightforward to express that some ordered pairs $mm$ code a surjective map of members of a plurality to its subpluralities.
Theorem (Plural Cantor’s theorem)

Let \( xx \) be a plurality with at least two members, and assume the domain has a pairing function. Then \( xx \) has more subpluralities than it has members in the precise sense that there is no plurality that codes a surjective mapping from \( xx \) to subpluralities of \( xx \).

Proof. Let \( x \equiv xx \) abbreviate (5), and let \( \delta(x) \) be the claim that \( x \) is not among any plurality that it codes, that is:

\[
\forall xx (x \equiv xx \rightarrow x \not\prec xx)
\]

I claim that \( \delta(x) \) is satisfied by at least one object. Consider two distinct objects \( a \) and \( b \). The claim follows by observing that there are three pluralities all of whose members are among \( a \) and \( b \), which are just two objects. Having established the claim, we can use plural comprehension to let \( dd \) be the 'diagonal plurality' comprising all and only the objects that satisfy \( \delta(x) \). Then we get \( d \prec dd \leftrightarrow d \not\prec dd \). \( \square \)
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Ascent Theorem (negative part)

Proposition

If absolute generality is possible, then a generalized semantics for a first-order language cannot be given in another first-order language.

Proof. Under the assumption of absolute generality, an ordinary singular predicate can be interpreted by means of any plurality. But by the generalization of Cantor's theorem, there are more pluralities than there are objects. It follows that interpretations of a first-order language cannot be objects but must be represented by means of higher-order resources. \(\Box\)
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Some generalized ascent theorems

Theorem (Generalized Ascent Theorem)

(a) A truth theory for an n-th order language $\mathcal{L}$ can be developed in the language that results from $\mathcal{L}$ by either adding an n-th order predicate or adding variables and quantifiers of order $n + 1$.
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(a) A truth theory for an $n$-th order language $\mathcal{L}$ can be developed in the language that results from $\mathcal{L}$ by either adding an $n$-th order predicate or adding variables and quantifiers of order $n + 1$.

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(c) A model theory for a general $n$-th order language $\mathcal{L}$ can be developed in a language of order $n+1$ with no non-logical constants.

(d) These results are optimal.
(Boolos, 1985) is a (re-)discovery of (a) for $n = 2$ on the plural interpretation of orders. (Rayo and Uzquiano, 1999) is a (re-)discovery of (b) and (c) for $n = 2$. (Rayo, 2006) proves the theorem in the generality stated above.
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The proofs are somewhat involved but can be broken down into components.

- (a) = (Tarski, 1935) + (Frege, 1879)/(Dedekind, 1888) + coding
- (b) = (Tarski, 1936) + (Frege, 1879)/(Dedekind, 1888) + coding
Transfinite generalizations

There is reason to think that Tarski knew all of this (and more):

\begin{itemize}
  \item[(a)] We cannot develop a generalized semantics for a language of order $\alpha$ in another language of order $\alpha$.
  \item[(b)] For any successor ordinal $\alpha$, we can develop a generalized semantics for a language of order $\alpha$ in a language of order $\alpha + 1$.
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In fact, the setting up of a correct definition of truth for languages of infinite order would in principle be possible provided we had at our disposal in the metalanguage expressions of higher order than all variables of the language investigated. (Tarski, 1935, p. 272)
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In fact, (Linnebo and Rayo, 2012) provide generalizations into the transfinite.

**Theorem (Transfinite Ascent Theorem)**

(a) We cannot develop a generalized semantics for a language of order $\alpha$ in another language of order $\alpha$.

(b) For any successor ordinal $\alpha$, we can develop a generalized semantics for a language of order $\alpha$ in a language of order $\alpha + 1$. 
- We need a variable assignment $A$, which assigns an entity of appropriate sort to each variable.

- When we do model theory, we also consider a model $M$ that specifies a domain and an assignment of appropriate entities to all constants of the language.

- Assume we can define the notions of an assignment and a model, as well as the assignment $\llbracket E \rrbracket = \llbracket E \rrbracket_{A,M}$ made to an expression $E$ relative to the variable assignment $A$ and model $M$.

- Assume the $n$-tuple of any entities can be coded by means of an entity whose order is no higher than that of the given entities.
We define the notion of satisfaction in a model $M$ relative to an assignment $A$ as follows:

1. If $\phi$ is a formula of the form $t(t_1, \ldots, t_n)$ for an $n$-place term $t$ and where the $t_i$ are of appropriate order, then:
   $$\text{Sat}\langle \lnot \phi \rangle, A \rangle \iff \langle \llbracket t_1 \rrbracket, \ldots, \llbracket t_n \rrbracket \rangle \mathcal{R} \llbracket t \rrbracket$$

2. If $\phi$ is a formula of the form $t_1 = t_2$ for two terms of the same order, then:
   $$\text{Sat}\langle \lnot \phi \rangle, A \rangle \iff \llbracket t_1 \rrbracket = \llbracket t_2 \rrbracket$$

3. If $\phi$ is a formula of the form $\lnot \psi$, then:
   $$\text{Sat}\langle \lnot \phi \rangle, A \rangle \iff \lnot \text{Sat}\langle \psi \rangle, A \rangle$$

4. If $\phi$ is a formula of the form $\psi_1 \land \psi_2$, then:
   $$\text{Sat}\langle \lnot \phi \rangle, A \rangle \iff \text{Sat}\langle \lnot \psi_1 \rangle, A \rangle \text{ and } \text{Sat}\langle \lnot \psi_2 \rangle, A \rangle$$

5. If $\phi$ is a formula of the form $\exists v \psi$ for some variable $v$, then:
   $$\text{Sat}\langle \lnot \phi \rangle, A \rangle \iff \text{Sat}\langle \lnot \psi \rangle, A \rangle \text{ for some variable } v$$
(Frege, 1879) and (Dedekind, 1888) discovered that recursive definitions can be turned into explicit ones by generalization over “collections” of the entities related by the recursive definition.

**Example**

Let $\text{Add}(x, y, z)$ ("$z$ is a sum of $x$ and $y$") be defined as follows:

- $\text{Add}(x, 0, x)$
- $\text{Add}(x, y, z) \rightarrow \text{Add}(x, y', z')$

Then $\text{Add}(x, y, z)$ can be defined explicitly as follows:

$$\forall R[\forall u R(u, 0, u) \land \forall u, v, w (R(u, v, w) \rightarrow R(u, v', w')) \rightarrow R(x, y, z)]$$
(Tarski, 1935) realized that his own recursive definition of satisfaction could be turned into an explicit one in this way.

- Replace $\text{Sat}(\lnot \phi, A)$ with $Y \langle \lnot \phi, A \rangle$ in each clause of the recursive definition (1).
- Conjoin the resulting clauses and abbreviate the resulting conjunction $\Phi(\lnot \phi, A)$.
- Then $\text{Sat}(\lnot \phi, A)$ can be defined explicitly as:

$$\text{Sat}(\lnot \phi, A) : \leftrightarrow \forall Y[\Phi(\lnot \phi, A) \rightarrow Y \langle \lnot \phi, A \rangle] \quad (6)$$
(Boolos, 1985) realizes that an assignment to plural variables can be taken to consist of some (set-theoretic) ordered pairs $aa$ of the form $\langle v, x \rangle$ where $v$ is a plural variable, subject to the requirement that for every plural variable $v$, there is at least one pair of the form $\langle v, x \rangle$ among $aa$.

We can even talk about the assignment $\llbracket v \rrbracket_{aa}$ made to the plural variable $v$ by $aa$:

$$\forall x [ x \not\in \llbracket v \rrbracket_{aa} \iff \langle v, x \rangle \not\in aa]$$

(7)

Likewise, it is straightforward to define the notion of a model.
Theorem \((n\text{-tuples})\)

Assume that for any two objects there is another object that serves as their ordered pair. Given any entities \(x^{k_1}, \ldots, x^{k_n}\), we can then code for the ordered \(n\)-tuple of these entities by means of a single entity \(x^k\), where \(k\) is the maximum of the \(k_i\). We will designate this entity \(x^k\) as \(\langle x^{k_1}, \ldots, x^{k_n} \rangle\).

**Proof idea.** The key step is to define the ordered pair of an object \(a\) and an \(n\)-th order entity \(x^n\). We proceed by induction. Assume we can have defined the ordered pair of \(a\) with any \(n\)-th order entity. Then we let the ordered pair \(\langle a, x^{n+1} \rangle\) of \(a\) with an entity \(x^{n+1}\) of order \(n + 1\) be the unique entity \(y^{n+1}\) such that:

\[
\forall u^n[y^{n+1}(u^n) \leftrightarrow \exists x^n(u^n = \langle a, x^n \rangle \land x^{n+1}(x^n))]
\]  

(8)
Proof sketch for Theorem 2 (b) and (c). Insert upper indices for orders in Definition 1 and use the three components to keep track of how high orders are needed to formulate the recursive definitions in questions and turn them into explicit ones. (For a detailed spelling out of this strategy, see (Linnebo and Rayo, 2012).)
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But we want a generalized semantics for this language too.
Review of the situation

- We want a generalized semantics for our first-order language.
- We can only get this by ascending to a second-order language.
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Review of the situation

- We want a generalized semantics for our first-order language.
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- So we ascend ...
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Rejecting the type theoretic defense of absolute generality

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Required by systematic theorizing

An extensionalist claims that any entity of any level is extensional

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