The problem of absolute generality - 1

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Some of our investigations and theories appear to require absolute generality, e.g. physicalism:

(3) Everything is physical.

Reasons to doubt the possibility of absolute generality (I)

But [the set-theoretic paradoxes] are only apparent 'contradictions', and depend solely on confusing set theory itself , which is not categorically determined by its axioms, with individual models representing it . What appears as an 'ultrafinite non- or super-set' in one model is, in the succeeding model, a perfectly good, valid set with both a cardinal number and an ordinal type, and is itself a foundation stone for the construction of a new domain. (Zermelo, 1930)

Reasons to doubt the possibility of absolute generality (II)

The argument from indefinite extensibility

There is a universal class X. But every class can, from an extended point of view, be regarded as a 'perfectly good, valid set'. Since sets are well-founded, the set corresponding to X cannot have been a member of X. So the universality of X was merely relative, not absolute.

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The semantic argument
Domains of quantification are sets. But standard set theory teaches us that there is no universal set. So no quantifier can range over absolutely everything, only over the elements of some set.

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- *Orthodox generality absolutism*: It is possible to quantify over absolutely everything. And there is nothing special about this range of quantification. It is just as definite as that of, say, quantification over electrons. (Boolos, 1985), (Cartwright, 1994), (Williamson, 2003)

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- *Alternative generality absolutism*: It is possible to quantify over absolutely everything. But the totality of absolutely everything is, in a certain sense, *indefinite*. (Dummett, 1991), (Linnebo, 2006), (Linnebo, 2010); anticipated by Cantor

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- Few, if any, philosophical views or arguments will be presupposed.
- All we presuppose is a willingness to reason outside of the (currently!) standard formal theories, such as ZFC. Such informal mathematical reasoning is legitimate and a necessary prerequisite to the formal reasoning.
- Most of all, we presuppose a willingness to think with an open mind about some hard logico-mathematical paradoxes.

Plan for the four lectures

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3. *Semantics and absolute generality*: the paradox of absolute generality; generality relativism dismissed; higher-order semantics
4. *Modal mathematics*: a modal explication of the notion of definiteness; modal set theory; property theory

Plural and higher-order logic (I)

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Second-order logic (SOL) allows us to generalize into the predicate position to conclude that there is a concept F under which Socrates falls:

$$\exists F\, F(\text{Socrates}) \tag{4}$$

Plural and higher-order logic (II)

Further combinations and extensions may be possible as well:

⋮	⋮	
SOL	PSO	...??
FOL	PFO	...??

The language and logic of SOL

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Deductive systems for SOL

- The usual I- and E-rules extended to the second-order quantifiers
- *Comprehension axioms* which specify what values the second-order variables can take:

$$\exists F \forall x[Fx \leftrightarrow \phi(x)] \tag{Comp}$$

where $\phi(x)$ does not contain F free.
- A second-order choice axiom can be added if desired, but we won't.

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- Same deductive system, except adjustments required by there being no empty plurality:
  - $\forall xx \exists u(u \prec xx)$
  - $\exists u\, \phi(u) \rightarrow \exists xx \forall u[u \prec xx \leftrightarrow \phi(u)]$

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Ontological innocence

- Plural logic is typically thought not to introduce any ontological commitments to sets, classes, reified 'pluralities', or the like.
- This will be particularly important when dealing with some of the problems surrounding absolute generality. For instance, 'the sets' refers to each and every set without commitment to any problematic universal set or class that mysteriously fails to be a set.

Paradoxes vs. antinomies

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An *antinomy* is a particularly stubborn paradox where we are unable to identify any culprit. That is, we are unable to reject one of the premises or to accept the conclusion—at least given the concepts involved.

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Why accept [Number totality]?

- Mathematicians do talk about sequences and 'collections' of numbers, e.g. *On*
- We can use plural or higher-order resources, e.g. talk about some number<u>s</u>, etc.

The paradox of numbers (Burali-Forti) (II)

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Why accept [Number extensibility 2]?

- Cantor showed that the sequence of natural numbers can be given a LUB. Why not other sequences of numbers too? A differential treatment would have to be justified!

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- Cantor showed that the sequence of natural numbers can be given a LUB. Why not other sequences of numbers too? A differential treatment would have to be justified!
- Merely to point to the threat of contradiction is 'to wield the big stick, not to offer an explanation' (Dummett, 1991, p. 316).

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But clearly, [Number totality] and [Number extensibility] are inconsistent.

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> *It is permissible to talk about 'collections' of sets (and other objects), including a collection of all sets.*

> **[Set extensibility]**
> *For any 'collection' of sets, there is another set.*

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- The iterative conception of set (Boolos, 1971)

The paradox of iterative sets (III)

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Why accept [Set extensibility 2]?

- Traditionally, most mathematicians and philosophers denied the existence of completed infinities, including infinite sets. But as Cantor realized, some infinite collections do give rise to sets. Why, then, should not all collections do so? What is the principled difference between collections that do and do not give rise to sets?

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- Part of the iterative conception: any 'available' objects can be used to form a set with precisely these objects as elements. And why shouldn't any 'collection' be 'available'?

How to respond to the paradoxes? (I)

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- **Extensibility**: Given any ‘collection’ of Fs, there is another F that is not in this ‘collection’.

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We call paradoxes with this structure *paradoxes of indefinite extensibility*.

Recall that, when reasoning about a paradox, it is dialectically unacceptable simply to use *reductio ad aburdum* to reject one particular assumption. Why is *this* the culprit rather than some other assumption?

The main responses: denying one of the premises

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Deny [Extensibility]?

Not all totalities characterized by means of plural or higher-order logic correspond to objects.

This is probably the most widespread response today. It will be critically examined in lecture 2.

The main responses: generality relativism

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This view does dispel the paradoxes. But the view is fraught with difficulties (Williamson, 2003), which we will seen in lecture 3.

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This response will be developed and examined in lecture 4.

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