The problem of absolute generality - 2

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[Number totality]
It is permissible to talk about sequences of numbers, including a sequence of all numbers.

[Number extensibility 1]
For any number α*, there is a successor* $\alpha + 1$.

[Number extensibility 2]
For any sequence of numbers, there is a least upper bound.

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Where we stand

- [Number totality] is rarely challenged when cashed out in terms of plural or higher-order logic.
- [Number extensibility 1] is rarely challenged.
- So let's focus on [Number extensibility 2]!

Slippery slope towards [Number extensibility 2]

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- If some infinite sequences of numbers have LUBs, why not all?

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Slippery slope arguments

- If some infinite sequences of numbers have LUBs, why not all?
- But perhaps there *is* a good answer!

From liberalism to [Number extensibility 2]

Mathematics is in its development entirely free and only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. (Cantor's Grundlagen, (Ewald, 1996, p. 896))

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- The generality relativist thinks so.
- But the generality absolutist will use her view to argue that the attempted definition isn't even coherent.
- So we're at a stand-off.

Can we do better?

The previous two arguments work, if at all, for *both* the plural *and* the second-order ways of cashing out the sequence talk.

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The argument from arbitrariness

- More compelling than the previous two arguments
- But works *only* for the plural analysis of the sequence talk
- Intuitive idea: For any plurality $\alpha\alpha$ of numbers, we can make good sense of there being more numbers. So it would be arbitrary if $\alpha\alpha$ were all the numbers there are.

The argument from arbitrariness

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- Assume that pure mathematical objects exist necessarily if at all.
- So it is possibly necessary that $\alpha\alpha$ have a LUB.
- So the desired conclusion follows by an application of B.

Defense of Step 1

Pluralities are modally rigid (Uzquiano, 2011):

$$x \prec xx \rightarrow \Box(Exx \rightarrow x \prec xx) \qquad (\Box \prec)$$

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So we can make perfect mathematical sense of a scenario in which $\alpha\alpha$ have a LUB. Indeed, we can produce a *mathematical model* of such a scenario. Thus:

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Contrast an intensionally characterized sequence, e.g. by means of the condition $On(x)$ of being an ordinal number. The analogue of (1) is the patently absurd claim:

$$\Diamond\exists\alpha\forall\beta(On(\beta) \leftrightarrow \beta < \alpha) \qquad (2)$$

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- Thus, if the sequence is given in extension, then either [Number extensibility 2] or an unattractive arbitrariness in pure mathematics.

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- Thus, if the sequence is given in extension, then either [Number extensibility 2] or an unattractive arbitrariness in pure mathematics.
- However, if the sequence cannot be given in extension but only in intension, it can only be tracked form world to world in terms of this intension.

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- Thus, if the sequence is given in extension, then either [Number extensibility 2] or an unattractive arbitrariness in pure mathematics.
- However, if the sequence cannot be given in extension but only in intension, it can only be tracked form world to world in terms of this intension.
- Then it is contradictory to assume that the sequence has a LUB.

Strategies for denying Extensibility (I)

Consider two parties who agree on a general theory of numbers but defend rival hypotheses concerning some particular numbers $\alpha\alpha$.

- H_0: $\alpha\alpha$ do not have a LUB but are all the numbers there are.
- H_1: $\alpha\alpha$ have a LUB and are thus not all the numbers there are.

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An easy victory?

- Whichever hypothesis is true is necessarily true.
- So if H_0 is true, then H_1 does not even represent a genuine possibility.
- But this is no more an argument against H_1 than it is an argument against atheism to insist that God exists by necessity.

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(i) limitation of size: that the sequence is small compared with the rest of the universe;

(ii) being generated by the iterative conception;

(iii) 'goodness' is a kind of 'definiteness', to be explicated in modal or other intensional terms.

(Slogan: The difference between sequences that have LUBs and ones that don't is *qualitative*, not merely *quantitative*.)

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Why should factors extrinsic to $\alpha\alpha$ determine whether $\alpha\alpha$ have a LUB?

- Had the universe been sufficiently populous, then $\alpha\alpha$ would had a LUB after all!
- There appear to be independent reasons to think that the universe might have been more populous than it actually is (Hawthorne and Uzquiano, 2011).

Circularity

The proposed explanation of why $\alpha\alpha$ fail to have a LUB is circular.

- We are told that H_1 is false because the cardinality of the universe does not permit it to be true.
- *But this cardinality assessment is made on the basis of H_0!*
- Had H_1 been true, then—as we will see—the cardinality of the universe would have been larger than it is according to H_0.
- In short, a circularity arises because the cardinality of the universe is part of what is to be explained and thus unsuitable to serve as the cornerstone of the explanation.

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We now turn to the details of the limitation of size proposal.

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So this yields: $\alpha\alpha$ have a LUB iff $\alpha\alpha$ have an upper bound.

Since it is common ground among the disputing parties that the ordinals are well-ordered, the proposed explanation of why some $\alpha\alpha$ do not have a LUB is useless.

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Proposition

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Proposition

Assume ($\star$). Then some numbers $\alpha\alpha$ are equinumerous with On iff $\alpha\alpha$ are cofinal in On.

We wanted an explanation of why H_0 is true rather than H_1. But the explanation we are offered is equivalent with H_0 *modulo* the agreed theory, assuming this includes ($\star$).

Proposition

Assume ($\star$). Then some numbers $\alpha\alpha$ are equinumerous with On iff $\alpha\alpha$ are cofinal in On.

Proof. Assume first that $\alpha\alpha$ are equinumerous with *On*. Then obviously the former are cofinal in the latter. We prove the converse by *reductio*. So assume $\alpha\alpha$ cannot be put in one-one correspondence with *On*. Consider the well-ordering based on $\alpha\alpha$ taken in their natural order. Since this well-ordering is shorter than that of *On*, there is a β such that $\alpha\alpha$ are equinumerous with the predecessors of β. Then it follows from the regularity of *On* that $\alpha\alpha$ cannot be cofinal in *On*. $\dashv$

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It remains to prove the claim.

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- Assume that for every well-ordered set exemplifies a number, and conversely, that every number is exemplified by a set.
- It follows that the cardinality of the universe is identical with the cardinality of the extended number sequence.

The iterative conception

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No surprise! A debate concerning the height of the cumulative hierarchy cannot be adjudicated simply by deferring to this hierarchy.

Cantor's attempt to deny [Extensibility]

Cantor's attempt is inspired by the ancient notion of *potential infinity*.

$$\Box \forall m \Diamond \exists n \, \textsc{Successor}(m, n) \tag{3}$$

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2. For any 'definite succession' of numbers, there is a least upper bound.

Thus, 'the extended number sequence' isn't definite.

Cantor on 'inconsistent multiplicities'

For a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call absolutely infinite or inconsistent multiplicities. [. . .]
If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a consistent multiplicity or a 'set'. (Ewald, 1996, p. 931-932)

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Cantor thought incompletability represents a *qualitative* difference between multiplicities that form sets and ones that do not, and that this qualitative difference *explains* why some but not all multiplicities are eligible for set formation.

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- How to understand the notion of definiteness or incompletability? Cantor says very little! We will be more explicit in lecture 4.

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- Today the orthodox view is to let $D(X)$ iff there is a set of all Xs. This is a purely extensional conception of definiteness. But the resulting view is unstable.
- Cantor's alternative proposal is to adopt an independent, intensional conception of definiteness.

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