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Lectures on Models of Arithmetic
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Lecture 2: Elementary Extensions

1. Theorem. *Every countable model $\mathcal{M}$ of* PA *has a proper elementary end extension.*
Proof: Let $\mathbb{B}$ be the Boolean algebra of all parametrically definable subsets of M, $\mathcal{F}$ be the collection of all parametrically definable functions from M to M, and $\mathcal{F}_{\text{bd}}$ be the collection of all $f \in \mathcal{F}$ such that the range of f is bounded in $\mathcal{M}$. Fix an enumeration $\langle f_n : n \in \omega \rangle$ of $\mathcal{F}_{\text{bd}}$.

It is not hard to construct $S_0 \supseteq S_1 \supseteq \cdots$ such that (a) $S_n \in \mathbb{B}$, (b) S_n is unbounded in M, and (c) f_n is constant on S_n. Let $\mathcal{U}_0$ be the Fréchet filter in the sense of $\mathcal{M}$. It is not hard to see $\{S_n : n \in \omega\} \cup \mathcal{U}_0$ uniquely extends to is a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{B}$.

Define $\sim$ on $\mathcal{F}$ via:

$$f \sim g \Longleftrightarrow \{m \in M : f(m) = g(m)\} \in \mathcal{U}.$$

Let $M^* := \mathcal{F}/\sim$. For $[f]$, $[g]$, and $[h]$ in M, define $+_M$ by

$$[f] +_M [g] = [h] \Longleftrightarrow \{m \in M : f(m) + g(m) = h(m)\} \in \mathcal{U}.$$

Similarly, one can define $\cdot_M$ and $<_M$. This gives rise to $\mathcal{M}^*$.

For each $m \in M$, let $c_m : M \to \{m\}$ be the constant m-function. This defines an embedding $m \longmapsto_j [c_m]$ from $\mathcal{M}$ into $\mathcal{M}^*$.

1.1. Łoś-style Theorem. *For any first order formula $\varphi(x_0, \cdots, x_{k-1})$ in the language of arithmetic, and any sequence $[f_0], \cdots, [f_{k-1}]$ from M^*, the following two conditions are equivalent:*

(a) $\mathcal{M}^* \models \varphi([f_0], \cdots, [f_{k-1}])$;
(b) $\{m \in M : \mathcal{M} \models \varphi(f_0(n), \cdots, f_{k-1}(n))\} \in \mathcal{U}$.

Proof of 1.1: Routine induction of the complexity of φ, except for the existential step case, where the "least number principle" is invoked. $\square$

Therefore the mapping j is an *elementary embedding*. Since the equivalence class of the identity function $i(m) = m$ is not in the range of j (since $\mathcal{U}_0 \subseteq \mathcal{U}$), this shows that $\mathcal{M}^*$ is a proper elementary extension of $\mathcal{M}$. To see that $\mathcal{M}^*$ *end extends* $\mathcal{M}$, suppose $\mathcal{M}^* \vDash [f] < [c_{m_0}]$ for some $m_0 \in M$. Then by the Łoś-style Theorem, we have

$$\overbrace{\{m \in M : \mathcal{M} \vDash f(m) < m\}}^{X} \in \mathcal{U}.$$

Let $f'(m) := f(m)$ if $m \in X$, and otherwise $f'(m) := 0$. Note that $[f'] = [f]$. Moreover, $f' \in \mathcal{F}_{\mathrm{bd}}$ and therefore $f' = f_k$ for some $k \in \omega$, which in turn implies (by design) that $f^{'}$ is constant on S_k with some value $m_1 \in M$. Hence $\mathcal{M}^* \vDash [f] = [c_{m1}]$. $\square$

2. Theorem. *The following scheme is provable in* PA (*and is known as the collection scheme*).

$$(\forall x < z \; \exists y \; \varphi(x, y, z)) \rightarrow (\exists v \; \forall x < z \; \exists y < v \; \varphi(x, y, z)) \, .$$

3. Theorem (Gaifman splitting, special case). *Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of* PA *with $\mathcal{M} \preceq \mathcal{N}$, and let $\overline{\mathcal{M}}$ be the submodel of $\mathcal{N}$ whose universe is the convex hull of $\mathcal{M}$ in $\mathcal{N}$. Then:*

$$\mathcal{M} \preceq_{\mathrm{cof}} \overline{\mathcal{M}} \preceq_{\mathrm{end}} \mathcal{N}.$$

Proof: It suffices to show that $\mathcal{M}^* \preceq \mathcal{N}$. We use the Tarski-test by supposing

$$\mathcal{N} \vDash \exists x \; \varphi(a, x),$$

where $a \in M^*$. Let $c \in M$ such that each $a_i < c$. Then, by invoking Collection in $\mathcal{M}$, there must be some $b \in M$ such that $\mathcal{M}$ satisfies the sentence

$$\forall z < c \; (\exists x \varphi(z, x) \rightarrow \exists x < b \; \varphi(z, x)) \, .$$

Since $\mathcal{N}$ satisfies the same sentence, this shows that we can find $c \in M^*$ such that $\mathcal{N} \models \varphi(a, c)$. $\Box$.

3.1. Corollary. *Every nonstandard model of* PA *has arbitrarily large cofinal extensions.*

- $\mathcal{N}$ is a *conservative* elementary extension of $\mathcal{M}$, written $\mathcal{M} \prec_{\text{cons}} \mathcal{N}$ if the intersection of any parametrically definable subset of $\mathcal{N}$ with M is also parametrically definable in $\mathcal{M}$.
- $\mathcal{N}$ is a *minimal* elementary extension of $\mathcal{M}$ if $\mathcal{M} \prec \mathcal{N}$ and the only elementary submodel of $\mathcal{N}$ properly extending $\mathcal{M}$ is $\mathcal{N}$ itself.

4. Proposition. *Conservative extensions of models of* PA *are end extensions.*

5. Theorem. *Every countable nonstandard model of* PA *has a proper cofinal minimal elementary extension.*

6. Open Problem (Problem 2 of Kossak-Schmerl). *Is there a nonstandard model of* PA *with no minimal elementary extension*?

7. Theorem. *Suppose* $\mathcal{L}$ *is a countable language extending* $\mathcal{L}_A$.
(a) (MacDowell-Specker 1959) *Every model* $\mathcal{M}$ *of* $\mathsf{PA}(\mathcal{L})$ *has a proper elementary end extension.*

(b) (Gaifman 1972, Phillips 1974) *In the above,* $\mathcal{N}$ *can be required to be both minimal and conservative extension of* $\mathcal{M}$.

- In what follows $\prod\limits_{\mathcal{U}} \mathcal{M}$ denotes the so-called Skolem (or definable) ultrapower obtained by considering *only functions from* M *to* M *that are parametrically definable in* $\mathcal{M}$

7.1. Lemma.
(a) ($\mathcal{M}$-completeness) $\mathcal{M} \prec_{\text{end}} \prod\limits_{\mathcal{U}} \mathcal{M}$ *iff for each* m *in* M, $M \to (\mathcal{U})^1_m$, *i.e., for any* $\mathcal{M}$*-parametrically definable* $f : M \to \{0, 1, \cdots, m-1\}$, f *is constant on a member of* $\mathcal{U}$.

(b) ($\mathcal{M}$-minimality) $\mathcal{M} \prec_{\text{min}} \prod\limits_{\mathcal{U}} \mathcal{M}$ *iff* $M \to (\mathcal{U})^2_2$, *i.e., for any* $\mathcal{M}$*-parametrically definable* $f : [M]^2 \to \{0, 1\}$, *some member* $X \in \mathcal{U}$ *is homogeneous for* f, *i.e.,* $|f([X]^2)| = 1$.

(c) ($\mathcal{M}$-iterability) $\mathcal{M} \prec_{\text{cons}} \prod_{\mathcal{U}} \mathcal{M}$ *is equivalent to each of the following:*

(c_1) *For each $\mathcal{M}$-parametrically definable $f : M \to M$, $\{m \in M : f^{-1}(m) \in \mathcal{U}\}$ is parametrically definable in $\mathcal{M}$.*

(c_2) *For each $\mathcal{M}$-parametrically definable $X \subseteq M$, and $m \in M$,*

$$(X)_m = \{x \in M : \langle m, x\rangle \in X\}.$$

(c_4) $M \to (\mathcal{U})^3_2$.

(c_5) $\forall m \in \omega \ \forall a \in M \ M \to (\mathcal{U})^m_a$.

It is known that for each fixed natural number n, Σ_n-truth is definable within $\mathcal{M}$. Therefore we can internally arrange all parametrically Σ_n-definable functions $f : M \to M$, as

$$\{f_n(x, m) : n < \omega, m \in M\}.$$

More specifically, for any $n < \omega$, and for any parameter m in M, $f_n(x, m)$ is defined by some Σ_n-formula $\psi(x, y, m)$ in $\mathcal{M}$, i.e.,

$$\forall a \forall b f_n(a, m) = b \textit{ iff } \mathcal{M} \models \psi(a, b, m)$$

Note that we can afford to use a single parameter m thanks to coding functions available in PA. Using both an *external* induction and an *internal* induction we will construct a doubly-indexed sequence of parametrically definable subsets of $\mathcal{M}$:

$$\{X_{m,n} : n \in \omega, \ m \in M\},$$

which can be arranged as the following $M \times \omega$ matrix:

$$\begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} & . & . & . & X_{0,n} & . & . & . \\ X_{1,0} & X_{1,1} & X_{1,2} & . & . & . & X_{1,n} & . & . & . \\ X_{2,0} & X_{2,1} & X_{2,2} & . & . & . & X_{2,n} & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ X_{m,0} & X_{m,1} & X_{m,2} & . & . & . & X_{m,n} & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

Suppose $\langle X_m : m \in M \rangle$ is a definable sequence of unbounded definable subsets of $\mathcal{M}$ with the property that $X_m \supseteq X_{m'}$ for $m < m'$. Even though $\bigcap_{m \in M} X_m$ might be empty, one can (internally) define an unbounded subset $\bigodot_{m \in M} X_m$ which is "almost contained" in each X_m, i.e., for each $m \in M$, the set $(\bigodot_{m \in M} X_m) \backslash X_m$ is $\mathcal{M}$-finite (i.e., bounded in $\mathcal{M}$). This is done by the following recursive definition within $\mathcal{M}$:

- $c_0 =$ the least member of X_0.
- $c_{m+1} =$ the first member of $\bigcap_{i \leq m} X_i$ which is greater than c_{m-1}.

Now let $\bigodot_{m \in M} X_m = \{c_m : m \in M\}$.

We can now construct our matrix column-by-column:

Let $X_{0,0} = M$ and suppose for some $k \geq 0$ we have constructed the $(k+1)^{th}$ column up to some "integer" J of $\mathcal{M}$, i.e., we have constructed $\langle X_{m,k} : m \leq J \rangle$ such that for each $m \leq J$, $X_{m,k}$is unbounded in $\mathcal{M}$ and $f_k(x, m)$ is either one-to-one or constant on $X_{m+1,k}$. Thanks to the provability of the formalized version of Ramsey's theorem in PA we can construct an unbounded definable unbounded subset $X_{J+1,m}$of $X_{J,m}$ on which $f_k(x, J)$ is one-to-one or constant. Assuming that for some $k < \omega$ the column $\langle X_{m,k} : m \in M \rangle$ has been constructed we begin the next column (by an *external* induction) by letting $X_{0,k+1} = \bigodot_{m \in M} X_{m,k}$.

Thus we have constructed a matrix of definable subsets of $\mathcal{M}$ satisfying the following conditions:

1. For each $n < \omega$ and $m \in M$, $X_{m,n}$is unbounded in $\mathcal{M}$ and $f_n(x, m)$ is either one-to-one or constant on $X_{m+1,n}$.
2. For each $n < \omega$ and $m \in M$, $X_{m',n} \subseteq X_{m,n}$ if $m \leq m'$. More generally: $X_{m',n'} \setminus X_{m,n}$ is finite in the sense of $\mathcal{M}$, provided $n < n'$ or ($n' = n$ and $m \leq m'$)..

Let $\mathcal{U}_1 := \{X_{m,n} : m \in M, n \in \omega\} \cup \{M \backslash \{0, 1, ..., m\} : m \in M\}$.

It is routine to verify that $\mathcal{U}_1$ generates a unique nonprincipal ultrafilter over the definable subsets of $\mathcal{M}$ such that the definable ultrapower $\prod_{\mathcal{U}} \mathcal{M}$ forms a *minimal conservative* elementary end extension of $\mathcal{M}$. □

3. In contrast with the usual construction of ultrapowers in general model theory where *all* functions from some index set I into the universe M of a model $\mathfrak{M}$ are used in the formation of the ultrapower, model theorists of arithmetic have found it useful to consider "limited" ultrapowers in which a manageable family of functions from I to M are selected to craft the ultrapower. The following three varieties (a), (b), and (c) of limited ultrapowers are the most well-known in the model theory of arithmetic:

(a) *Skolem-Gaifman ultrapowers*, where the index set I is identical to the universe M of the model $\mathcal{M}$, and the family of functions used in the formation of the ultrapower is the set of all $\mathcal{M}$-definable ones. This sort of ultrapower was implicitly used by Skolem in his original construction of a nonstandard model of arithmetic, and they were employed by MacDowell-Specker in the proof of their celebrated theorem. Later, in the work of Gaifman, Skolem ultrapowers were refined to a high degree of sophistication to produce a variety of striking results. One of Gaifman's key insights was that the Skolem ultrapower construction can be iterated along *any* linear order with appropriately chosen ultrafilters.

(b) *Kirby-Paris ultrapowers,* where the index is a regular cut I of $\mathcal{M}$, and the family of functions used in the formation of the ultrapower are functions f such that for some function g coded in $\mathcal{M}$, $f = g \restriction I$. This has proved to a valuable tool in the study of cuts of nonstandard models of arithmetic.

(c) *Paris-Mills ultrapowers,* where the index set is some topped initial segment of $\mathcal{M}$, and the functions used are those that are coded in $\mathcal{M}$. This type of ultrapower was first considered by Paris and Mills to show, among other things, that one can arrange a model of PA in which an externally countable nonstandard integer H such that the external cardinality of $\mathsf{Superexp}(2, H)$ is of any prescribed infinite cardinality. Here $\mathsf{Superexp}(x, y)$ is the result of y iterations of the exponential function 2^x.

- Suppose

  (a) $\mathcal{M} = (M, \cdots)$ is a structure,

  (b) $\mathcal{U}$ is an ultrafilter over $\mathcal{P}(\omega)$, and

  (c) $\mathbb{L}$ is a linear order.

  One can build the $\mathbb{L}$-iterated ultrapower of $\mathcal{M}$ modulo $\mathcal{U}$.

$$\mathcal{M}^* := \prod_{\mathcal{U}, \mathbb{L}} \mathcal{M}.$$

- A key definition (reminiscent of Fubini):

$$\mathcal{U}^2 := \{X \subseteq \omega^2 : \{a \in \omega : \overbrace{\{b \in \omega : (a, b) \in X\}}^{(X)_a} \in \mathcal{U}\} \in \mathcal{U}.$$

- More generally, for each nonzero $n \in \omega$:

$$\mathcal{U}^{n+1} := \{X \subseteq \omega^{n+1} : \{a \in \omega : (X)_a \in \mathcal{U}^n\} \in \mathcal{U}\},$$

  where

$$(X)_a := \{(b_1, \cdots, b_n) : (a, b_1, \cdots, b_n) \in X\}.$$

  Let Υ be the set of terms τ of the form $f(l_1, \cdots, l_n)$, where $n \in \omega$, $f : \omega^n \to M$ and $(l_1, \cdots, l_n) \in [\mathbb{L}]^n$.

- The universe M^* of $\mathcal{M}^*$ consists of equivalence classes $\{[\tau] : \tau \in \Upsilon\}$, where the equivalence relation $\sim$ at work is defined as follows: given $f(l_1, \cdots, l_r)$ and $g(l'_1, \cdots, l'_s)$ from Υ, first suppose that

$$\left(l_1, \cdots, l_r, l'_1, \cdots, l'_s\right) \in [\mathbb{L}]^{r+s};$$

  let $p := r + s$, and define:

$$f(l_1, \cdots, l_r) \sim g(l'_1, \cdots, l'_s)$$

  iff

$$\{(i_1, \cdots, i_p) \in \omega^p : f(i_1, \cdots, i_r) = g(i_{r+1}, \cdots, i_p)\} \in \mathcal{U}^p.$$

More generally: given $f(l_1, \cdots, l_r)$ and $g(l_1', \cdots, l_s')$ from Υ, let

$$P := \{l_1, \cdots, l_r\} \cup \{l_1', \cdots, l_s'\}, \quad p := |P|,$$

and relabel the elements of P in increasing order as $\bar{l}_1 < \cdots < \bar{l}_p$. This relabelling gives rise to increasing sequences $(j_1, j_2, \cdots, j_r)$ and $(k_1, k_2, \cdots, k_s)$ of indices between 1 and p such that

$$l_1 = \bar{l}_{j_1}, l_2 = \bar{l}_{j_2}, \cdots, l_r = \bar{l}_{j_r}$$

and

$$l_1' = \bar{l}_{k_1}, l_2' = \bar{l}_{k_2}, \cdots, l_s' = \bar{l}_{k_s}.$$

With the relabelling at hand, we can define:

$$f(l_1, \cdots, l_r) \sim g(l_1', \cdots, l_s')$$

iff

$$\{(i_1, \cdots, i_p) \in \omega^p : f(i_{j_1}, \cdots, i_{j_r}) = g(i_{k_1}, \cdots, i_{k_s})\} \in \mathcal{U}^p.$$

- We can also use the previous relabelling to define other operations and relations of $\mathcal{M}^*$

- For $m \in M$, let c_m be the constant m-function on ω, i.e., $c_m : \omega \to \{m\}$. For any $l \in \mathbb{L}$, we can identify the element $[c_m(l)]$ with m.

- We shall also identify $[id(l)]$ with l, where $id : \omega \to \omega$ is the identity function (WLOG $\omega \subseteq M$).

- Therefore $M \cup \mathbb{L}$ can be viewed as a subset of M^*.

7.2. Theorem (Gaifman)**.** *For every formula $\varphi(x_1, \cdots, x_n)$, and every $(l_1, \cdots, l_n) \in [\mathbb{L}]^n$, the following two conditions are equivalent:*

(a) $\mathcal{M}^* \models \varphi(l_1, l_2, \cdots, l_n)$.

(b) $\{(i_1, \cdots, i_n) \in \omega^n : \mathcal{M} \models \varphi(i_1, \cdots, i_n)\} \in \mathcal{U}^n$.

- The above construction can be miniaturized in a number of contexts, including the 3 types of ultrapowers mentioned earlier. More on this topic, in tomorrow's lecture.