Nordic Spring School in Logic 2013
Sophus Lie Conference Center, Nordfjordeid
Lectures on Models of Arithmetic
Ali Enayat, University of Gothenburg

Lecture 1: Classical Results

1. Models of arithmetic can be described as structures that share many first order properties of the structure $\mathbb{N} := (\omega, +, \cdot, S(x), <, 0)$, or some *expansion* of $\mathbb{N}$.

- The language of first order arithmetic is $\{\overbrace{+, \cdot, \mathsf{S}(x), <, 0}^{\mathcal{L}_A}, \cdots\}$.
- Models of arithmetic are structures of the form

$$\mathcal{M} := (M,\ +_M,\ \cdot_M,\ <_M,\ 0_M,\ 1_M, \cdots).$$

1.1. The most well-known axiom systems of arithmetic are the following:

Q, IOpen, $\mathsf{I\Delta_0}$, $\mathsf{I\Delta_0} + \Omega_1, \cdots, \mathsf{I\Delta_0} + \mathsf{Exp}, \cdots, \mathsf{PRA}$, $\cdots$, $\mathsf{I\Sigma_1}, \cdots, \mathsf{PA}, \cdots$, $\mathsf{A^{ZF}}$.

- Q consists of the universal generalizations of the following axioms:

  (i) $\mathsf{S}(x) \neq 0$
  (ii) $(\mathsf{S}(x) = \mathsf{S}(y)) \rightarrow (x = y)$
  (iii) $(y = 0) \vee \exists x\ (\mathsf{S}(x) = y)$
  (iv) $(x + 0) = x$
  (v) $(x + \mathsf{S}(y)) = \mathsf{S}(x + y)$
  (vi) $(x \cdot 0) = 0$
  (vii) $(x \cdot \mathsf{S}(y)) = ((x \cdot y) + x)$
  (viii) $(x < y) \leftrightarrow \exists z\,((x + \mathsf{S}(z)) = y)$

- Given a family Φ of $\mathcal{L}_A$-formulae, $\mathsf{I}\Phi$ consists of Q plus universal generalizations of formulae of the following form, where $\varphi \in \Phi$.

$$(\varphi(0,\overline{z}) \wedge \forall x\ \ (\varphi(x,\overline{z}) \rightarrow \varphi(\mathsf{S}(x),\overline{z}))) \rightarrow \forall x\ \varphi(y,\overline{z}).$$

- Open := quantifier-free $\mathcal{L}_A$-formulae.
- $\Sigma_0 := \Pi_0 := \Delta_0 :=$

$$\{\exists x < y\ \varphi : \varphi \in \mathsf{Open}\} \cup \{\forall x < y\ \varphi : \varphi \in \mathsf{Open}\}.$$

- $\Sigma_{n+1} := \{\exists x_0 \cdots \exists x_{n-1}\ \varphi : \varphi \in \Pi_n\}$.
- $\Pi_{n+1} := \{\forall x_0 \cdots \forall x_{n-1}\ \varphi : \varphi \in \Sigma_n\}$.
- Ω_1 := "$2^{|x|^2}$ is a total function", where $|x|$ is the length of the binary expansion of x.
- Exp := "2^x is a total function".[1]
- PRA := Primitive Recursive Arithmetic.
- A^{ZF} := "what ZF knows about arithmetic".

1.2. $\mathrm{Th}(\mathbb{N})$ is known as *true arithmetic.*

1.3. $\mathrm{Th}(\mathbb{N}, X)_{X\subseteq\omega}$, is commonly known as *full arithmetic*. The study of models of full arithmetic is closely related to the study of ultrafilters on $\mathcal{P}(\omega)$.

[1]By a classical theorem of Parikh $I\Delta_0$ can only prove the totality of functions with a polynomial growth rate, hence

$$I\Delta_0 \nvdash \forall x \exists y \varphi(x,y).$$

However, the graph of the exponential function $y = 2^x$ can be defined by a Δ_0-predicate in the standard model of arithmetic (a nontrivial fact). Indeed, it is known that there are Δ_0-predicates $\varphi(x,y)$ which do the job, and have the property that $I\Delta_0$ can prove the familiar algebraic laws about exponentiation for $\varphi(x,y)$. It is now known that the graphs of many other fast growing recursive functions, such as the superexponential function, the Ackermann function, and indeed all functions $\{F_\alpha : \alpha < \varepsilon_0\}$ in the (fast growing) Wainer hierarchy, can be defined by Δ_0-predicates for which $I\Delta_0$ can prove appropriate recursion schemes.

1.4. Our focus in this series of talks will be on models of PA (but the results that go through for weaker systems will be pointed out). Our meta-theory is ZFC, but practically all of the results in these lectures can be established in second order order arithmetic. Note that the consistency of PA is a theorem of our meta-theory.

2. $\mathcal{M}$ is *nonstandard* if $(M, <_M) \ncong (\omega, <)$. Equivalently, $\mathcal{M}$ is nonstandard if there is an element $c \in M$ such that for all $n \in \omega$,

$$c >_M (\overbrace{1 +_M 1 +_M \cdots +_M 1}^{n \text{ times}}).$$

2.1. Theorem (Skolem 1934) *There is a a nonstandard model of true arithmetic.*

Proof: (1) Let $\mathbb{B}$ be the Boolean algebra of all arithmetical sets, i.e., subsets of ω that are definable by some arithmetical formula. Fix an enumeration $\{X_n : n \in \omega\}$ of $\mathbb{B}$.

(2) Let $\mathcal{U}_0$ be the Fréchet filter, and define S_n via the following recursive definition . We say that a family of sets has the finite intersection property (f.i.p.) if the intersection of any finite number of elements of the family is nonempty.

$$S_n = \begin{cases} X_n & , \mathcal{U}_0 \cup \{S_i : i < n\} \cup \{X_n\} \text{ has f.i.p.} \\ \omega \backslash X_n & , \text{otherwise.} \end{cases}$$

(3) It is easy to see that $\{S_n : n \in \omega\}$ is a nonprincipal ultrafilter $\mathcal{U}$ over $\mathbb{B}$.

(4) Let $\mathcal{F}$ be the collection of all functions from ω to ω whose graph is arithmetical. Define $\sim$ on $\mathcal{F}$ via:

$$f \sim g \iff \{n \in \omega : f(n) = g(n)\} \in \mathcal{U}.$$

(5) Let $M := \mathcal{F}/\sim$. For $[f]$ and $[g]$ in M, define $+_M$ by

$$[f] +_M [g] = [h] \iff \{n \in \omega : f(n) + g(n) = h(n)\} \in \mathcal{U}.$$

Similarly, one can define $\cdot_M$ and $<_M$.

(6) For each $n \in \omega$, let $c_n : \omega \to \{n\}$ be the constant n-function. This defines an embedding $n \longmapsto_j [c_n]$ from $\mathbb{N}$ into $\mathcal{M}$.

(7) **Łoś-style Theorem**. *For any first order formula $\varphi(x_0, \cdots, x_{k-1})$ in the language of arithmetic, and any sequence $[f_0], \cdots, [f_{k-1}]$ from M, the following two conditions are equivalent:*

(a) $\mathcal{M} \models \varphi([f_0], \cdots, [f_{k-1}])$;
(b) $\{n \in \omega : \mathcal{M} \models \varphi(f_0(n), \cdots, f_{k-1}(n))\} \in \mathcal{U}$.

Proof: Routine induction of the complexity of φ, except for the existential step case, where the "least number principle" is invoked.

(8) Therefore the mapping j (defined in (6) above) is an *elementary embedding*. Since the equivalence class of the identity function $i(n) = n$ is not in the range of j, this shows that $\mathcal{M}$ is a nonstandard model of $Th(\mathbb{N})$. This concludes the proof of Skolem's theorem.

2.2. Theorem (Ryll-Nardzewski 1952, Rabin 1962) *No consistent extension of* PA *is axiomatizable by sentences of bounded complexity.*

3. Theorem (Gödel 1931, Rosser 1936) *There are* $2^{\aleph_0}$ *distinct completions of* PA.

4. Theorem (Folklore). *There are* $2^{\aleph_0}$ *pairwise nonisomorphic countable models of any completion* T *of* PA.

Key idea of the proof: Let p_n is the n-th prime. For each $A \subseteq \omega$, the following 1-type is $\Sigma_A(v)$ is consistent with T

$$\Sigma_A(v) := \{v \mid p_n : n \in A\} \cup \{v \nmid p_n : n \notin A\}.$$

4.1. With more work, one can even show that for each infinite cardinal θ, there are 2^θ pairwise nonisomorphic models of each completion of PA . [2]

[2] This can be done by using Gaifman machinery's of minimal types (plus the existence of 2^θ pairwise nonisomorphic linear orders). Moreover, by a theorem of Shelah, any countable first order theory with an infinite model that has a definable linear ordering has maximal number of models in each infinite cardinality.

5. Theorem (Henkin 1950). *The order type of nonstandard models of* PA *is of the form*

$$\omega + \mathbb{Z}\mathbb{L},$$

Where $\mathbb{L}$ *is a dense linear order without endpoints. Therefore the order type of countable nonstandard models of* PA *is of the form*

$$\omega + \mathbb{Z}\mathbb{Q}.$$

5.1. Open Problem (Problem 14 of Kossak-Schmerl). *Suppose* $\theta \geq \aleph_1$, *and* T_1 *and* T_2 *be any two completions of* PA. *Is it true that the class of order types of models of* T_1 *of cardinality* θ *coincide with the class of order types of models of* T_2 *of cardinality* θ?

6. The arithmetized version of the completeness theorem can be used to show that there is a nonstandard model $\mathcal{M}$ of PA of the form

$$\mathcal{M} := (\omega, +_M, \cdot_M, <_M, 0_M, 1_M)$$

such that Th($\mathcal{M}$) is Δ_2, i.e., Turing reducible to $0'$. [3]

7. Theorem (Ackermann, 1940). *Let* E *be defined*[4] *by:*

$$aEb \text{ iff the } a\text{-th digit of the binary expansion of } b \text{ is } 1.$$

Then $(\omega, E) \cong (V_\omega, \in)$, *where* V_ω *is the collection of hereditarily finite sets.*[5]

7.1. Moreover, if $\mathcal{M} \models$ PA, then $(M, E) \models \mathsf{ZF}^{-\infty}$ + "every set has a transitive closure".[6]

---

[3]Moreover, the *Low Basis Theorem* of Jockusch and Soare can be used to improve Δ_2 to "low-Δ_2", i.e., a Δ_2-set X such that $X' \leq_T 0'$.

[4]With some work, E can be arrange to be a Δ_0-formula.

[5]Recall that that $(V_\omega, \in)$ is a model of $\mathsf{ZF}^{-\infty} := \mathsf{ZF} \backslash \{\mathsf{Infinity}\} \cup \{\neg \mathsf{Infinity}\}$ + "every set has a transitive closure".

[6]Indeed, PA is *bi-interpretable* with $\mathsf{ZF}^{-\infty}$ + "every set has a transitive closure, but not with $\mathsf{ZF}^{-\infty}$.

8. Let $\mathcal{M}$ be a nonstandard model of PA, and $c \in M$.

$$c_E := \{x \in M : xEc\}$$

$$\mathrm{SSy}(\mathcal{M}) := \{c_E \cap \omega : c \in M\}.$$

8.1. Theorem (Folklore). *Let $\mathcal{M}$ be a nonstandard model of* PA. *Then* $\mathrm{SSy}(\mathcal{M})$ *coincides with the collection of "traces" of parameterically definable subset of $\mathcal{M}$ on ω, i.e.,*

$$\mathrm{SSy}(\mathcal{M}) = \{D \cap \omega : D \textit{ is a parametrically definable subset of } \mathcal{M}\}.$$

8.2. Theorem (Scott 1962). *A countable family $\mathcal{S}$ of subsets of ω is of the form* $\mathrm{SSy}(\mathcal{M})$ *for some model of* PA *iff it is a "Scott set", i.e.,*

(i) *$\mathcal{S}$ is a Boolean algebra,*

(ii) *$\mathcal{S}$ is closed under Turing reducibility, and*

(iii) *Every countable infinite subtree of $2^{<\omega}$ that is coded in $\mathcal{S}$, has an infinite branch that is a member of $\mathcal{S}$.*

8.3. It is known[7] that Scott's theorem is also true for families $\mathcal{S}$ of size $\aleph_1$.

8.4. Open Problem (Problem 1 of Kossak-Schmerl). *What is the status of Scott's theorem for the absence of the continuum hypothesis?*

8.5. Theorem. *Suppose $\mathcal{M} := (\omega, +_M, \cdot_M, \cdots)$ is a nonstandard model of arithmetic.*

(a) (Tennenbaum 1959) *If $\mathcal{M} \models \mathsf{PA}$, then $+_M$ nor $\cdot_M$ is recursive.*

(b) (Feferman 1960) *If $\mathcal{M} \models \mathsf{TA}$, then neither $+_M$ nor $\cdot_M$ is arithmetical.*

Proof outline:

(1) Every member of $\mathrm{SSy}(\mathcal{M})$ is recursive in each of $+_M$ and $\cdot_M$.

(2) If $\mathcal{M}$ is a model of PA, then $\mathrm{SSy}(\mathcal{M})$ contains nonrecursive sets; and if $\mathcal{M}$ is a model of TA, then $\mathrm{SSy}(\mathcal{M})$ contains all arithmetical sets.

8.6. Theorem (Mostowski). *The set $\mathsf{Val}_{\mathrm{Rec}}$ of first order sentences that are true in all recursive (computable) structures is not even arithmetically definable.*

---

[7] This was independently established by a number of researchers, including Ehrenfeucht, Guaspari, Friedman, and Knight-Nadel.

Proof outline: the proof of Tennenbaum's theorem goes through for models of $\mathsf{I}\Sigma_1$, which is known to be axiomatizable by a single sentence σ. Therefore, for any arithmetical sentence ψ,

$$(\sigma \wedge \psi) \in \mathsf{Val}_{\mathrm{Rec}} \Longleftrightarrow \mathbb{N} \models \psi.$$

Therefore TA is Turing reducible to $\mathsf{Val}_{\mathrm{Recursive}}$, so we are done by Tarski's undefinability of truth theorem.

8.7. In contrast, many interesting fragments of arithmetic have recursive nonstandard models, for example, by a theorem of Shepherdson (1964), IOpen *has recursive nonstandard models.*[8]

8.8. Theorem (Shepherdson 1964). *Let R be a discrete ordered ring and let K be the real closure of the fraction field of R. R is a model of IOpen iff $\forall a \in K\ \exists r \in R$ such that $|r - a| < 1$.*

9. Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of $\mathcal{L}_A$, where $\mathcal{M} \subseteq \mathcal{N}$.

(a) $\mathcal{N}$ *end extends* $\mathcal{N}$, written $\mathcal{M} \subseteq_{\mathrm{end}} \mathcal{N}$, if $a <_N b$ for every $a \in M$, and $b \in N \backslash M$.

(b) $\mathcal{N}$ *cofinally extends* $\mathcal{M}$, written $\mathcal{M} \subseteq_{\mathrm{cof}} \mathcal{N}$, if for each $b \in N$ there is some $a \in M$ such that $a <_N b$.

9.1. Theorem (MacDowell-Specker Theorem 1961). *Every model of* PA *has an elementary end extension.*

9.2. Theorem (Gaifman 1972). *Suppose $\mathcal{M}$ and $\mathcal{N}$ are models of* PA *with $\mathcal{M} \subseteq \mathcal{N}$, and let $\overline{\mathcal{M}}$ be the submodel of $\mathcal{N}$ whose universe is the convex hull of $\mathcal{M}$ in $\mathcal{N}$. Then:*

$$\mathcal{M} \preceq_{\mathrm{cof}} \overline{\mathcal{M}} \subseteq_{\mathrm{end}} \mathcal{N}.$$

9.3. Theorem (Gaifman 1976). *Suppose* $\mathcal{M} \models \mathsf{PA}$.
(a) *There is a proper e.e.e. $\mathcal{N}$ of $\mathcal{M}$ that possesses an automorphism j such that* $\mathrm{Fix}(j) = M$.
(b) *For every linear order $\mathbb{L}$, there is an elementary end extension $\mathcal{N}_{\mathbb{L}}$ of $\mathcal{M}$ such that* $\mathrm{Aut}_M(\mathcal{N}_{\mathbb{L}}) \cong \mathrm{Aut}(\mathbb{L})$.

10. Theorem (Friedman 1973). *Every countable model of* PA *is isomorphic with a proper initial segment of itself.*

[8]The work of Wilkie, Smith, Berrarducci, Otero, Moniri, and Mohsenipour has unearthed stronger extensions of IOpen that have recursive models.

References

[1] P. Hájek and P. Pudlák, **Metamathematics of First Order Arithmetic**, Springer, Heidelberg, 1993.

[2] R. Kaye, **Models of Arithmetic**, Oxford Logic Guides, vol. 15, Oxford University Press, 1991.

[3] R. Kossak and J. Schmerl, **The Structure of Models of Arithmetic**, Oxford University Press, 2006.

[4] K. McAloon, **Models of Arithmetic and Complexity Theory**, Studies in complexity theory, 119-221, Res. Notes Theoret. Comput. Sci., Pitman, London, 1986.

[5] C. Smoryński, *Course Notes on models of arithmetic* (*Utrecht 1978*), *available at*

`http://igitur-archive.library.uu.nl/ph/2010-1214-200237/preprint289.pdf`

[6] ___________, *Lectures on models of arithmetic,* in **Logic Colloquium '82** (edited by G. Lolli and A. Marcja), North Holland, Amsterdam, 1984.