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- III. Classification of Provable Functions and Functionals of Theories
- IV. Combinatorial Independence Results

Examples

- I. (R; Setzer) Consistency proof of $(\Sigma^1_2\text{-}\mathbf{AC}) + \mathbf{BI}$ in Martin-Löf Type Theory.

- A *finite tree* is a finite partially ordered set

$$\mathbb{B} = (B, \leq)$$

such that:

(i) B has a smallest element (called the *root* of $\mathbb{B}$);

(ii) for each $s \in B$ the set $\{t \in B : \ t \leq s\}$ is a totally ordered subset of B.

Combinatorial Independence Results

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- For finite trees $\mathbb{B}_1$ and $\mathbb{B}_2$, an embedding of $\mathbb{B}_1$ into $\mathbb{B}_2$ is a one-to-one mapping

$$f : \mathbb{B}_1 \to \mathbb{B}_2$$

  such that

$$f(a \wedge b) = f(a) \wedge f(b)$$

  for all $a, b \in \mathbb{B}_1$, where $a \wedge b$ denotes the infimum of a and b.

- **Kruskal's Theorem.** For every infinite sequence of trees $(\mathbb{B}_k : k < \omega)$, there exist i and j such that $i < j < \omega$ and $\mathbb{B}_i$ is embeddable into $\mathbb{B}_j$.
(In particular, there is no infinite set of pairwise nonembeddable trees.)

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- **Theorem** (H. Friedman, D. Schmidt) Kruskal's Theorem is not provable in $\mathbf{ATR}_0$.

- The proof utilizes that Kruskal's Theorem implies that Γ_0 is well-founded.

The Extended Kruskal Theorem

- For $n < \omega$, let $\mathcal{B}_n$ be the set of all finite trees with labels from n, i.e. $(\mathbb{B}, \ell) \in \mathcal{B}_n$ if $\mathbb{B}$ is a finite tree and

$$\ell : B \to \{0, \ldots, n-1\}.$$

The set $\mathcal{B}_n$ is quasiordered by putting $(\mathbb{B}_1, \ell_1) \leq (\mathbb{B}_2, \ell_2)$ if there exists an embedding

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$$f : \mathbb{B}_1 \to \mathbb{B}_2 \qquad \text{such that:}$$

- $\ell_1(b) = \ell_2(f(b))$ for each $b \in B_1$;
- if b is an immediate successor of $a \in \mathbb{B}_1$, then for each $c \in \mathbb{B}_2$ in the interval $f(a) < c < f(b)$,

$$\ell_2(c) \geq \ell_2(f(b)).$$

This condition is called a **gap condition**.

The Extended Kruskal Theorem

Theorem. (Friedman) For each $n < \omega$, $\mathcal{B}_n$ is a **well quasi ordering** (abbreviated WQO($\mathcal{B}_n$)), i.e. there is no infinite set of pairwise nonembeddable trees.

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Theorem $\forall n < \omega$ WQO($\mathcal{B}_n$) is not provable in $\Pi^1_1 - \mathbf{CA}_0$.

- The proof employs an ordinal representation system for the proof-theoretic ordinal of $\Pi^1_1 - \mathbf{CA}_0$.
  The ordinal is $\psi_0(\Omega_\omega)$.

The Graph Minor Theorem

- $\mathbb{G}$, $\mathbb{H}$ graphs. If $\mathbb{H}$ is obtained from $\mathbb{G}$ by first deleting some vertices and edges, and then contracting some further edges, $\mathbb{H}$ is a **minor** of $\mathbb{G}$.

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  GMT Theorem. (Robertson and Seymour 1986-1997) If $\mathbb{G}_0, \mathbb{G}_1, \mathbb{G}_2, \ldots$ is an infinite sequence of finite graphs, then there exist $i < j$ so that $\mathbb{G}_i$ is isomorphic to a minor of $\mathbb{G}_j$.

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- **Corollary.** (Vázsonyi's conjecture) If all the $\mathbb{G}_k$ are trivalent, then there exist $i < j$ so that $\mathbb{G}_i$ is embeddable into $\mathbb{G}_j$.

- **Corollary.** (Wagner's conjecture) For any 2-manifold M there are only finitely many graphs which are not embeddable in M and are minimal with this property.

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  - GMT is not provable in $\Pi^1_1 - \mathbf{CA}_0$.

Ockham's Razor

In what follows, we shall be solely dealing with classical logic. Therefore we can simplify the sequent calculus as follows:

- We get rid of the **structural rules** by using **sets** of formulae rather than **sequents** of formulae. This has the effect that **exchange** and **contraction** happen automatically:

  $$\{C_1, \ldots, C_r, A, A, D_1, \ldots, D_s\} = \{D_1, \ldots, D_s, A, C_1, \ldots, C_r\}$$

  We take care of **weakening** by adding all the formulae we may be interested in from the start; thus we have more liberal axioms:

  $$A, \Gamma \Rightarrow \Delta, A$$

- Using the **De Morgan laws** of classical logic we can push **negations** in front of **atomic** formulae. Also, in classical logic $\neg, \wedge, \vee$ forms a **complete** set of connectives. Thus we can simplify matters, by demanding that formulae are built up from **atomic** and **negated atomic formulae** (literals) by means of $\wedge, \vee, \forall, \exists$.
  Negating a formula A then becomes a defined operation:
  - $\neg\neg A := A$ if A is atomic;
  - $\neg(A \wedge B) = \neg A \vee \neg B$; $\neg(A \vee B) = \neg A \wedge \neg B$;
  - $\neg\forall x\, F(x) := \exists x\, \neg F(x)$; $\neg\exists x\, F(x) := \forall x\, \neg F(x)$.
- In classical logic we don't need the two sides of a sequent

$$A_1, \ldots, A_r \Rightarrow \Delta$$

since it can be re-written as

$$\Rightarrow \neg A_1, \ldots, \neg A_r, \Delta$$

In the Tait-style version of the **classical sequent calculus** $\Gamma, \Delta, \Lambda, \Theta, \ldots$ range over finite sets of formulae in **negation normal form.** Γ, Δ stands for $\Gamma \cup \Delta$ and Δ, A is short for $\Delta \cup \{A\}$.

The **inferences** of the Tait-calculus are as follows:

(Axiom) $\Gamma,\ A,\ \neg A$

($\wedge$) $$\frac{\Gamma, A \qquad \Gamma, A'}{\Gamma, A \wedge A'}$$

($\vee$) $$\frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1$$

($\forall$) $$\frac{\Gamma,\ F(a)}{\Gamma,\ \forall x\, F(x)}$$

($\exists$) $$\frac{\Gamma,\ F(t)}{\Gamma,\ \exists x\, F(x)}$$

(Cut) $$\frac{\Gamma,\ A \qquad \Gamma,\ \neg\, A}{\Gamma}$$

s an **eigenvariable**.

Part II: Predicative Proof Theory

Ramified Analysis $\mathbf{RA}_\alpha$

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Theorem: (Schütte)

The proof-theoretic ordinal of $\mathbf{RA}_\alpha$ *is* $\varphi\alpha 0$.

Infinite Ramified Analysis $\mathbf{RA}^{\infty}$

Uses ordinal indexed free set variables $U^{\alpha}, V^{\alpha}, W^{\alpha}, \ldots$ and and bound set variables $X^{\beta}, Y^{\beta}, Z^{\beta}, \ldots$ with $\beta > 0$.

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3. If A and B are formulas of levels α and β, then $A \vee B$ and $A \wedge B$ are formulas of level $\max(\alpha, \beta)$.

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4. If $F(0)$ is a formula of level α, then $\forall x F(x)$ and $\exists x F(x)$ are formulas of level α and $\{x \mid F(x)\}$ is a **set term** of level α.

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5. If $F(U^\beta)$ is a formula of level α and $\beta > 0$, then $\forall X^\beta F(X^\beta)$ is a formula of level $\max(\alpha, \beta)$.

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5. $|\forall X^\alpha A(X^\alpha)| = |\exists X^\alpha A(X^\alpha)| = \max(\omega \cdot \gamma, |A(U^0)| + 1)$

where γ is the level of $\forall X^\alpha A(X^\alpha)$.

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Cut-elimination II:

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- Π^1_1-$\mathbf{CA}_0$ is an impredicative theory.

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- Gödel's Constructible Hierarchy **L**:

$$\begin{aligned} \mathbf{L}_0 &= \emptyset, \\ \mathbf{L}_\lambda &= \bigcup\{\mathbf{L}_\beta : \beta < \lambda\} \ \lambda \text{ limit} \\ \mathbf{L}_{\beta+1} &= \{X : X \subseteq \mathbf{L}_\beta;\ X \text{ definable over } \langle \mathbf{L}_\beta, \in \rangle\}. \end{aligned}$$

The **axioms** of **KP** are:

Extensionality:	$a = b \rightarrow [F(a) \leftrightarrow F(b)]$
Foundation:	$\exists x G(x) \rightarrow \exists x[G(x) \wedge (\forall y \in x)\neg G(y)]$
Pair:	$\exists x\ (x = \{a, b\})$.
Union:	$\exists x\ (x = \bigcup a)$.
Infinity:	$\exists x\ \left[x \neq \emptyset\ \wedge\ (\forall y \in x)(\exists z \in x)(y \in z)\right]$.
Δ_0 Separation:	$\exists x\ \left(x = \{y \in a : F(y)\}\right)$
	$F(y)$ Δ_0-formula.
Δ_0 Collection:	$(\forall x \in a)\exists y G(x, y) \rightarrow \exists z(\forall x \in a)(\exists y \in z)G(x, y)$
	for all Δ_0–formulas G.

By a Δ_0 formula we mean a formula of set theory in which all the quantifiers appear restricted, that is have one of the forms $(\forall x \in b)$ or $(\exists x \in b)$.

We have seen that in the case of **PA** the addition of an infinitary rule enables us to regain cut elimination.

ω–rule:

$$\frac{\Gamma, A(\bar{n}) \text{ for all } n}{\Gamma, \forall x\, A(x)}.$$

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An ordinal analysis for **PA** is then attained as follows:

- Each **PA**–proof can be "unfolded" into a $\mathbf{PA}_\omega$–proof of the same sequent.
- Each such $\mathbf{PA}_\omega$–proof can be transformed into a cut–free $\mathbf{PA}_\omega$–proof of the same sequent of length $< \varepsilon_0$.

In order to obtain a similar result for set theories like **KP**, we have to work a bit harder. Guided by the ordinal analysis of **PA**, we would like to invent an infinitary rule which, when added to **KP**, enables us to eliminate cuts.

The first ordinal analysis of **KP** was given by Jäger in 1978.

As opposed to the natural numbers, it is not clear how to bestow a canonical name to each element of the set–theoretic universe.

Here we will use Gödel's constructible universe L. The constructible universe is “made" from the ordinals. It is pretty obvious how to “name” sets in L once we have names for ordinals at our disposal.

Recall that L_α, the αth level of **Gödel's constructible hierarchy** L, is defined by

$$\begin{aligned} L_0 &= \emptyset, \\ L_\lambda &= \bigcup\{L_\beta : \beta < \lambda\} \ \lambda \text{ limit} \\ L_{\beta+1} &= \{X : X \subseteq L_\beta;\ X \text{ definable over } \langle L_\beta, \in\rangle\}. \end{aligned}$$

So any element of L of level α is definable from elements of L with levels $< \alpha$ and the parameter L_{α_0} if $\alpha = \alpha_0 + 1$.

- Henceforth Ω will be a name for a large ordinal or even the whole class of ordinals.

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- The problem of "naming" sets will be solved by building a formal constructible hierarchy using the ordinals $< \Omega$.

Definition The RS_Ω–terms and their levels are generated as follows.

1. For each $\alpha < \Omega$,

$$\mathbb{L}_\alpha$$

   is an RS_Ω–term of level α.

2. The formal expression

$$\{x \in \mathbb{L}_\alpha : F(x, \vec{s})^{\mathbb{L}_\alpha}\}$$

   is an RS_Ω–term of level α if $F(a, \vec{b})$ is an $\mathcal{L}$–formula (whose free variables are among the indicated) and $\vec{s} \equiv s_1, \cdots, s_n$ are RS_Ω–terms with levels $< \alpha$.

   $F(x, \vec{s})^{\mathbb{L}_\alpha}$ results from $F(x, \vec{s})$ by restricting all unbounded quantifiers to $\mathbb{L}_\alpha$.

Let $\mathcal{T}$ be the collection of all RS_Ω-terms.
For $t \in \mathcal{T}$, $|t|$ denotes the level of t, i.e. the maximum ordinal α such that $\mathbb{L}_\alpha$ occurs in t.

We denote by upper case Greek letters

$$\Gamma, \Delta, \Lambda, \ldots$$

finite sets of RS_Ω–formulae. The intended meaning of

$$\Gamma = \{A_1, \cdots, A_n\}$$

is the disjunction

$$A_1 \vee \cdots \vee A_n$$

Γ, A stands for $\Gamma \cup \{A\}$ etc..

The rules of RS_Ω are:

$$(\wedge) \quad \frac{\Gamma, A \quad \Gamma, A'}{\Gamma, A \wedge A'}$$

$$(\vee) \quad \frac{\Gamma, A_i}{\Gamma, A_0 \vee A_1} \quad \text{if } i = 0 \text{ or } i = 1$$

$$(b\forall) \quad \frac{\cdots \Gamma, s \in t \rightarrow F(s) \cdots (|\,s\,| < |\,t\,|)}{\Gamma, (\forall x \in t) F(x)}$$

$$(b\exists) \quad \frac{\Gamma, s \in t \wedge F(s)}{\Gamma, (\exists x \in t) F(x)} \quad \text{if } |\,s\,| < |\,t\,|$$

$$(\forall) \quad \frac{\cdots \Gamma,\ F(s) \cdots (s \in \mathcal{T})}{\Gamma, \forall x\, F(x)}$$

$$(\exists) \quad \frac{\Gamma,\ F(s)}{\Gamma, \exists x\, F(x)} \quad \text{if } s \in \mathcal{T}$$

$$(\notin) \qquad \frac{\cdots \Gamma, s \in t \to r \neq s \cdots\cdots (|\, s \,| < |\, t \,|)}{\Gamma, r \notin t}$$

$$(\in) \qquad \frac{\Gamma, s \in t \ \wedge \ r = s}{\Gamma, r \in t} \quad \text{if } |\, s \,| < |\, t \,|$$

$$(\mathrm{Cut}) \qquad \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}$$

$$(\mathrm{Ref}_{\Sigma}) \qquad \frac{\Gamma, A}{\Gamma, \exists z\, A^z} \quad \text{if } A \text{ is a } \Sigma\text{-formula,}$$

where a formula is said to be in Σ if all its unbounded quantifiers are existential.
A^z results from A by restricting all unbounded quantifiers to z.

$\mathcal{H}$–controlled derivations

If we dropped the rule (Ref_Σ) from RS_Ω, the remaining calculus would enjoy full cut elimination owing to the symmetry of the pairs of rules

$$
\begin{array}{cc}
(\wedge) & (\vee) \\
(\forall) & (\exists) \\
(\notin) & (\in)
\end{array}
$$

However, partial cut elimination for RS_Ω can be attained by delimiting a collection of derivations of a very uniform kind. Buchholz developed a very elegant and flexible setting for describing uniformity in infinitary proofs, called operator controlled derivations.

Definition Let

$$P(ON) = \{X : X \text{ is a set of ordinals}\}.$$

A class function

$$\mathcal{H} : P(ON) \to P(ON)$$

will be called operator if $\mathcal{H}$ is a closure operator, i.e monotone, inclusive and idempotent, and satisfies the following conditions for all $X \in P(ON)$:

1. $0 \in \mathcal{H}(X)$.
2. If α has Cantor normal form $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, then $\alpha \in \mathcal{H}(X) \iff \alpha_1, ..., \alpha_n \in \mathcal{H}(X)$.

The latter ensures that $\mathcal{H}(X)$ will be closed under $+$ and $\sigma \mapsto \omega^\sigma$, and decomposition of its members into additive and multiplicative components.

For a term s, the operator $\mathcal{H}[s]$ is defined by

$$\mathcal{H}[s](X) \quad = \quad \mathcal{H}(X \cup \{\text{ all ordinals in } s\})$$

Definition Let $\mathcal{H}$ be an operator and let Γ be a finite set of RS_Ω–formulae.

$$\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$$

is defined by recursion on α. It is always demanded that

$$\{\alpha\} \cup k(\Gamma) \subseteq \mathcal{H}(\emptyset).$$

The inductive clauses are:

$$(\exists) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, F(s)}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \exists x F(x)} \quad \begin{array}{r} \alpha_0 < \alpha \\ |s| < \alpha \end{array}$$

$$(\forall) \quad \frac{\mathcal{H}[s] \overset{\alpha_s}{\underset{\rho}{\vdash}} \Gamma, F(s) \text{ for all } s}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \forall x F(x)} \quad |s| \leq \alpha_s < \alpha$$

$$(\mathrm{Cut}) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, B \qquad \mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, \neg B}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma} \quad \begin{array}{r} \alpha_0 < \alpha \\ rk(B) < \rho \end{array}$$

$$(Ref_\Sigma) \quad \frac{\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, A}{\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma, \exists z\, A^z} \quad \begin{array}{r} \alpha_0, \Omega < \alpha \\ A \in \Sigma \end{array}$$

To connect **KP** with the infinitary system RS_Ω one has to show that **KP** can be embedded into RS_Ω. Indeed, the finite **KP**-derivations give rise to very uniform infinitary derivations.

Theorem:

If

$$\mathbf{KP} \vdash B(a_1, \ldots, a_r)$$

then

$$\mathcal{H} \,\Big|\frac{\Omega \cdot m}{\Omega + n}\, B(s_1, \ldots, s_r)$$

holds for some m, n *and all set terms* $s_1, \ldots, s_r$ *and operators* $\mathcal{H}$ *satisfying*

$$\{\xi \;:\; \xi \text{ occurs in } B(\vec{s})\} \;\subseteq\; \mathcal{H}(\emptyset).$$

m and n depend only on the **KP***-derivation of* $B(\vec{a})$.

The usual cut elimination procedure works as long as the cut formulae have not been introduced by an inference Ref_{Σ}. As the principal formula of an inference Ref_{Σ} has rank Ω one gets the following result.

Theorem: (Cut elimination I)

$$\mathcal{H} \;\frac{\;\alpha\;}{\Omega+n+1}\; \Gamma \quad \Rightarrow \quad \mathcal{H} \;\frac{\;\omega_n(\alpha)\;}{\Omega+1}\; \Gamma$$

where $\omega_0(\beta) := \beta$ *and* $\omega_{k+1}(\beta) := \omega^{\omega_k(\beta)}$.

The obstacle to pushing cut elimination further is exemplified by the following scenario:

$$\dfrac{\dfrac{\mathcal{H} \Big|\!\frac{\delta}{\Omega}\, \Gamma, A}{\mathcal{H} \Big|\!\frac{\xi}{\Omega}\, \Gamma, \exists z\, A^z}(\mathrm{Ref}_\Sigma) \qquad \dfrac{\cdots \mathcal{H}[s] \Big|\!\frac{\xi_s}{\Omega}\, \Gamma, \neg A^s \cdots (s \in \mathcal{T})}{\mathcal{H} \Big|\!\frac{\xi}{\Omega}\, \Gamma, \forall z\, \neg A^z}(\forall)}{\mathcal{H} \Big|\!\frac{\alpha}{\Omega+1}\, \Gamma}(Cut)$$

Fortunately, it is possible to eliminate cuts in the above situation provided that the side formulae Γ are of complexity Σ. The technique is known as “collapsing” of derivations.

In the course of “collapsing” one makes use of a simple bounding principle.

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Lemma: (Boundedness)

Let A *be a* Σ*-formula,* $\alpha \leq \beta < \Omega$*, and* $\beta \in \mathcal{H}(\emptyset)$*. If*

$$\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, A$$

then

$$\mathcal{H} \,\Big|\!\frac{\alpha}{\rho}\, \Gamma, A^{\mathbb{L}_\beta}\,.$$

If the length of a derivation of Σ-formulae is $\geq \Omega$, then "collapsing" results in a shorter derivation, however, at the cost of a much more complicated controlling operator.

An ordinal representation system for the Bachmann-Howard ordinal

The **Veblen-function** φ figures prominently in elementary proof theory.

It is defined by transfinite recursion on α by letting $\varphi_0(\xi) := \omega^\xi$ and, for $\alpha > 0$, φ_α be the function that enumerates the class of ordinals

$$\{\gamma : \forall \xi < \alpha\, [\varphi_\xi(\gamma) = \gamma]\}.$$

We shall write $\varphi\alpha\beta$ instead of $\varphi_\alpha(\beta)$.

Let Γ_α be the α^{th} ordinal $\rho > 0$ such that for all $\beta, \gamma < \rho$, $\varphi\beta\gamma < \rho$.

Corollary

1. $\varphi 0\beta = \omega^\beta$.
2. $\xi, \eta < \varphi\alpha\beta \Longrightarrow \xi + \eta < \varphi\alpha\beta$.
3. $\xi < \zeta \Longrightarrow \varphi\alpha\xi < \varphi\alpha\zeta$.
4. $\alpha < \beta \Longrightarrow \varphi\alpha(\varphi\beta\xi) = \varphi\beta\xi$.

The least ordinal (> 0) closed under the function φ is known as

$$\Gamma_0$$

The proof-theoretic ordinal of **KP**, however, is bigger than Γ_0 and we need another function to obtain a sufficiently large ordinal representation system.

Let Ω be a "big" ordinal. By recursion on α we define sets $C^{\Omega}(\alpha, \beta)$ and the ordinal $\psi_{\Omega}(\alpha)$ as follows:

$$C^{\Omega}(\alpha, \beta) = \begin{cases} \textit{closure of } \beta \cup \{0, \Omega\} \\ \textit{under:} \\ +, (\xi \mapsto \omega^{\xi}) \\ (\xi \longmapsto \psi_{\Omega}(\xi))_{\xi < \alpha} \end{cases} \tag{2}$$

$$\psi_{\Omega}(\alpha) \simeq \min\{\rho < \Omega : C^{\Omega}(\alpha, \rho) \cap \Omega = \rho\}. \tag{3}$$

Note that if $\psi_\Omega(\alpha)$ is defined, then

$$\psi_\Omega(\alpha) < \Omega$$

and

$$[\psi_\Omega(\alpha), \Omega) \; \cap \; C^\Omega(\alpha, \psi_\Omega(\alpha)) \; = \; \emptyset$$

thus the order-type of the ordinals below Ω which belong to the Skolem hull $C^\Omega(\alpha, \psi_\Omega(\alpha))$ is $\psi_\Omega(\alpha)$.

In more pictorial terms, $\psi_\Omega(\alpha)$ is the α^{th} collapse of Ω.

Lemma $\psi_\Omega(\alpha)$ is always defined; in particular $\psi_\Omega(\alpha) < \Omega$.

Proof: The claim is actually not a definitive statement as I haven't yet said what largeness properties Ω has to satisfy. In the proof below, we assume $\Omega := \aleph_1$, i.e. Ω is the first uncountable cardinal.

Observe first that for a limit ordinal λ,

$$C^{\Omega}(\alpha, \lambda) = \bigcup_{\xi<\lambda} C^{\Omega}(\alpha, \xi)$$

since the right hand side is easily shown to be closed under the clauses that define $C^{\Omega}(\alpha, \lambda)$.

Now define

$$\begin{aligned} \eta_0 &= \sup C^\Omega(\alpha, 0) \cap \Omega \\ \eta_{n+1} &= \sup C^\Omega(\alpha, \eta_n) \cap \Omega \\ \eta^* &= \sup_{n<\omega} \eta_n. \end{aligned} \tag{4}$$

Since for $\eta < \Omega$ the cardinality of $C^\Omega(\alpha, \eta)$ is the same as that of $\max(\eta, \omega)$ and therefore less than Ω, the regularity of Ω implies that $\eta_0 < \Omega$. By repetition of this argument one obtains $\eta_n < \Omega$, and consequently $\eta^* < \Omega$. The definition of η^* then ensures

$$C^\Omega(\alpha, \eta^*) \cap \Omega \;=\; \bigcup_n C^\Omega(\alpha, \eta_n) \cap \Omega \;=\; \eta^* < \Omega.$$

Therefore, $\psi_\Omega(\alpha) < \Omega$. □

Let

$$\varepsilon_{\Omega+1}$$

be the least ordinal $\alpha > \Omega$ such that $\omega^\alpha = \alpha$.

The next definition singles out a subset

$$\mathcal{T}(\Omega)$$

of

$$C^\Omega(\varepsilon_{\Omega+1}, 0)$$

which gives rise to an ordinal representation system, i.e., there is an elementary ordinal representation system

$$\langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle$$

so that

$$\langle \mathcal{T}(\Omega), <, \Re, \psi, \ldots \rangle \;\cong\; \langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle. \tag{5}$$

"…" is supposed to indicate that more structure carries over to the ordinal representation system.

Definition $\mathcal{T}(\Omega)$ is defined inductively as follows:

1. $0, \Omega \in \mathcal{T}(\Omega)$.
2. If $\alpha_1, \ldots, \alpha_n \in \mathcal{T}(\Omega)$ and $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} > \alpha_1 \geq \ldots \geq \alpha_n$, then $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \in \mathcal{T}(\Omega)$.
3. If $\alpha \in \mathcal{T}(\Omega)$ and $\alpha \in C^{\Omega}(\alpha, \psi_{\Omega}(\alpha))$, then $\psi_{\Omega}(\alpha) \in \mathcal{T}(\Omega)$.

The side condition in the second clause is easily explained by the desire to have unique representations in $\mathcal{T}(\Omega)$.

The requirement

$$\alpha \in C^{\Omega}(\alpha, \psi_{\Omega}(\alpha))$$

in the third clause also serves the purpose of unique representations (and more) but is probably a bit harder to explain. The idea here is that from $\psi_{\Omega}(\alpha)$ one should be able to retrieve the stage (namely α) where it was generated. This is reflected by

$$\alpha \in C^{\Omega}(\alpha, \psi_{\Omega}(\alpha)).$$

It can be shown that the foregoing definition of $\mathcal{T}(\Omega)$ is deterministic, that is to say every ordinal in $\mathcal{T}(\Omega)$ is generated by the inductive clauses in exactly one way. As a result, every

$$\gamma \in \mathcal{T}(\Omega)$$

has a unique representation in terms of symbols for

$$0, \Omega$$

and function symbols for

$$+, \alpha \mapsto \omega^{\alpha}, \ \alpha \mapsto \psi_{\Omega}(\alpha).$$

The unique representation of will be referred to as the normal form.

Thus, by taking some primitive recursive (injective) coding function $\lceil \cdots \rceil$ on finite sequences of natural numbers, we can code $\mathcal{T}(\Omega)$ as a set of natural numbers as follows:

$$\ell(\alpha) = \begin{cases} \lceil 0, 0 \rceil & \text{if } \alpha = 0 \\ \lceil 1, 0 \rceil & \text{if } \alpha = \Omega \\ \lceil 2, \ell(\alpha_1), \cdots, \ell(\alpha_n) \rceil & \text{if } \alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \\ \lceil 3, \ell(\beta), \ell(\Omega) \rceil & \text{if } \alpha = \psi_\Omega(\beta), \end{cases}$$

where the distinction by cases refers to the unique representation of ordinals in $\mathcal{T}(\Omega)$. With the aid of ℓ, the ordinal representation system (5) can be defined by letting $\mathcal{OR}$ be the image of ℓ and setting

$$\lhd \ := \ \{(\ell(\gamma), \ell(\delta)) : \gamma < \delta \wedge \delta, \gamma \in \mathcal{T}(\Omega)\}$$

etc. However, a proof that this definition of

$$\langle \mathcal{OR}, \lhd, \hat{\Re}, \hat{\psi}, \ldots \rangle$$

in point of fact furnishes an elementary ordinal representation system is a bit lengthy.

Theorem: (Collapsing Theorem)

Let Γ be a set of Σ-formulae. Then we have

$$\mathcal{H}_\eta \mid\frac{\alpha}{\Omega+1} \Gamma \quad \Rightarrow \quad \mathcal{H}_{f(\eta,\alpha)} \mid\frac{\psi_\Omega(f(\eta,\alpha))}{\psi_\Omega(f(\eta,\alpha))} \Gamma$$

where $(\mathcal{H}_\xi)_{\xi \in \mathcal{T}(\Omega)}$ is a uniform sequence of ever stronger operators.

$$\mathcal{H}_\delta(X) = \bigcap\{C^\Omega(\alpha,\beta) :\ X \subseteq C^\Omega(\alpha,\beta)\ \wedge\ \delta < \alpha\}$$

From the Bounding Lemma it follows that all instances of Ref_{Σ} can be removed from derivations of length $< \Omega$.

For derivations without instances of Ref_{Σ} there is predicative cut-elimination.

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Theorem: (Predicative cut elimination)

$$\mathcal{H} \,\Big|\!\frac{\delta}{\rho}\, \Gamma \ \textit{and}\ \delta, \rho < \Omega \quad \Rightarrow \quad \mathcal{H} \,\Big|\!\frac{\varphi\rho\delta}{0}\, \Gamma \,.$$

The ordinal $\psi_\Omega(\varepsilon_{\Omega+1})$ is known as the Bachmann-Howard ordinal. Combining the previous results of this section, one obtains:

Corollary: If A is a Π_2-formula and

$$\mathbf{KP} \vdash A$$

then

$$L_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A.$$

The bound of this Corollary is sharp, that is, $\psi_\Omega(\varepsilon_{\Omega+1})$ is the first ordinal with that property.

Power Kripke-Platek Set Theory

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We call a formula of $\mathcal{L}_\in$ $\Delta_0^{\mathcal{P}}$ if all its quantifiers are of the form $Q\,x \subseteq y$ or $Q\,x{\in}y$ where Q is $\forall$ or $\exists$ and x and y are distinct variables.

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The $\Delta_0^{\mathcal{P}}$ formulas are the smallest class of formulae containing the atomic formulae closed under $\wedge, \vee, \rightarrow, \neg$ and the quantifiers

$$\forall x \in a,\ \exists x \in a,\ \forall x \subseteq a,\ \exists x \subseteq a.$$

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KP($\mathcal{P}$) has the following axioms: Extensionality, Pairing, Union, Infinity, Powerset, $\Delta_0^{\mathcal{P}}$-Separation and $\Delta_0^{\mathcal{P}}$-Collection.

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2. Alternatively, **KP**($\mathcal{P}$) can be obtained from **KP** by adding a function symbol $\mathcal{P}$ for the powerset function as a primitive symbol to the language and the axiom

   $$\forall y\,[y \in \mathcal{P}(x) \leftrightarrow y \subseteq x]$$

   and extending the schemes of Δ_0 Separation and Collection to the Δ_0 formulae of this new language.

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   and extending the schemes of Δ_0 Separation and Collection to the Δ_0 formulae of this new language.
3. The **power admissible** sets are the transitive models of **KP**($\mathcal{P}$).

The techniques used for the ordinal analysis of **KP** can be adapted to yield the following result about $\mathbf{KP}(\mathcal{P}) + \mathbf{AC}$:

Theorem (R. 2012)
If A is a $\Pi_2^{\mathcal{P}}$-formula and

$$\mathbf{KP}(\mathcal{P}) + \mathbf{AC} \vdash A$$

then

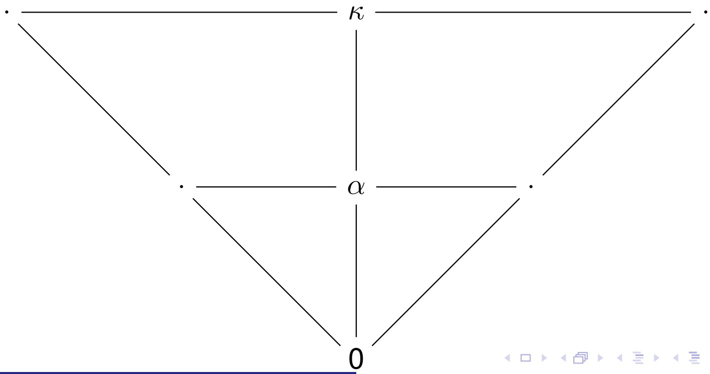
$$V_{\psi_\Omega(\varepsilon_{\Omega+1})} \models A.$$

The bound of this Corollary is sharp, that is, $\psi_\Omega(\varepsilon_{\Omega+1})$ is the first ordinal with that property.

Π_2-*Reflection*

Admissible sets $\mathbf{L}_\kappa$ have the property that if $A(x, y)$ is a bounded formula, then

$$\mathbf{L}_\kappa \models \forall x\, \exists y\, A(x, y) \quad \Rightarrow \quad \exists \alpha < \kappa\; \mathbf{L}_\alpha \models \forall x\, \exists y\, A(x, y).$$



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- **KPi** $\equiv \Delta^1_2$-**CA** $+$ **BI**

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- **KP** $+ \Sigma_1$-Separation and (Π^1_2-**CA**) $+$ **BI** prove the same sentences of second order arithmetic.
- An admissible ordinal κ is nonprojectible if there is no total κ–recursive function mapping κ one–one into some $\beta < \kappa$, where a function $g : \mathbf{L}_\kappa \to \mathbf{L}_\kappa$ is called κ–recursive if it is Σ_1 definable in $\mathbf{L}_\kappa$.

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- $\mathbf{L}_\kappa \models \mathbf{KP} + \Sigma_1$-Separation iff κ is non-projectible.

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- $\mathbf{KP} + \Sigma_1$-Separation and $(\Pi^1_2\text{-}\mathbf{CA}) + \mathbf{BI}$ prove the same sentences of second order arithmetic.
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- $\mathbf{L}_\kappa \models \mathbf{KP} + \Sigma_1$-Separation iff κ is non-projectible.
- κ is nonprojectible iff $\mathbf{L}_\kappa$ is a limit of Σ_1–elementary substructures, i.e. for every $\beta < \kappa$ there exists $\beta < \rho < \kappa$ such that $\mathbf{L}_\rho \prec_1 \mathbf{L}_\kappa$.

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$$\mathbf{L}_{\rho+\delta} \models A[\rho, a_1, \ldots, a_n],$$

- then there exists $\rho_0, \delta_0 < \rho$ and $b_1, \ldots, b_r \in \mathbf{L}_{\rho_0+\delta_0}$ such that

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- Conjecture: Π^1_3-Comprehension is the generic case.

The End

THANK YOU!

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Und wenn du lange in einen Abgrund blickst, blickt der Abgrund auch in dich hinein.

And if you gaze for long into an abyss, the abyss gazes also into you.

Friedrich NIETZSCHE (1886) *Jenseits von Gut und Böse*