PROOF THEORY:
From arithmetic to set theory

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Plan of First and Second Talk

- **The origins of Proof theory: Hilbert's Programme**
- **Gentzen's Result**
- **The General Form of Ordinal Analysis**
- **Gentzen's Hauptsatz: Cut Elimination**
- **A Brief History of Ordinal Representation Systems**
- **A Brief History of Ordinal Analyses**
- **Applications of Ordinal Analysis**
  1. **Combinatorial Independence Results**
  2. **Classification of Provable Functions**
  3. **Equiconsistency Results**

Plan of the Third and Fourth Talk

PREDICATIVE PROOF THEORY

IMPREDICATIVE PROOF THEORY

- **Ordinal Analysis of Kripke-Platek Set Theory (sketch)**
- **Uniformity of Infinite Proofs**
- **Proof Theory of Much Stronger Theories**

The Origins of Proof Theory (Beweistheorie)

- Hilbert's second problem (1900): Consistency of Analysis
- Hilbert's Programme (1922,1925)

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- To carry out this task, Hilbert inaugurated a new mathematical discipline: **Beweistheorie** (**Proof Theory**).
- In Hilbert's Proof Theory, proofs become mathematical objects sui generis.

Ackermann's Dissertation 1925

Consistency proof for a second-order version of **Primitive Recursive Arithmetic**.

Uses a finitistic version of **transfinite induction** up to the ordinal ω^{ω^ω}.

Gentzen's Result

- **Gerhard Gentzen** showed that transfinite induction up to the ordinal

$$\varepsilon_0 \;=\; \mathsf{sup}\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\} \;=\; \mathsf{least}\ \alpha.\ \omega^\alpha = \alpha$$

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- Gentzen's applied transfinite induction up to ε_0 solely to **primitive recursive predicates** and besides that his proof used only **finitistically justified means**.

Gentzen's Result in Detail

- $$\mathbf{F} + \text{PR-}\mathbf{TI}(\varepsilon_0) \vdash \mathbf{Con}(\mathbf{PA}),$$

  where **F** signifies a theory that is acceptable in finitism (e.g. $\mathbf{F} = \mathbf{PRA}$ = Primitive Recursive Arithmetic) and PR-**TI**(ε_0) stands for transfinite induction up to ε_0 for primitive recursive predicates.

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- Gentzen also showed that his result is best possible: **PA** proves transfinite induction up to α for arithmetic predicates for any $\alpha < \varepsilon_0$.

The **non-finitist** part of **PA** is encapsulated in PR-**TI**(ε_0) and therefore "measured" by ε_0, thereby tempting one to adopt the following definition of **proof-theoretic ordinal** of a theory T:

$$|T|_{Con} \quad = \quad \text{least } \alpha.\ \mathbf{PRA} + \text{PR-}\mathbf{TI}(\alpha) \vdash \text{Con}(T).$$

The supremum of the provable ordinals

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- The supremum of the provable well-orderings of **T**:

$$|\mathbf{T}|_{\mathrm{sup}} := \sup\{\alpha : \alpha \text{ provably computable in } \mathbf{T}\}.$$

Ordinal Structures

We are interested in representing specific ordinals α as relations on $\mathbb{N}$.

Natural ordinal representation systems are frequently derived from structures of the form

$$\mathfrak{A} \quad = \quad \langle \alpha, f_1, \ldots, f_n, <_\alpha \rangle$$

where α is an ordinal, $<_\alpha$ is the ordering of ordinals restricted to elements of α and the f_i are functions

$$f_i : \underbrace{\alpha \times \cdots \times \alpha}_{k_i \text{ times}} \longrightarrow \alpha$$

for some natural number k_i.

$$\mathbb{A} \quad = \quad \langle A, g_1, \ldots, g_n, \prec \rangle$$

is a **computable** (or **recursive**) **representation** of $\mathfrak{A} = \langle \alpha, f_1, \ldots, f_n, <_\alpha \rangle$ if the following conditions hold:

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2. $\prec$ is a computable total ordering on A and the functions g_i are computable.
3. $\mathfrak{A} \cong \mathbb{A}$, i.e. the two structures are isomorphic.

Cantor's Representation of Ordinals

Theorem (Cantor, 1897) For every ordinal $\beta > 0$ there exist unique ordinals $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ such that

$$\beta = \omega^{\beta_0} + \ldots + \omega^{\beta_n}. \tag{1}$$

The representation of β in (1) is called the **Cantor normal form**.

We shall write $\beta =_{CNF} \omega^{\beta_1} + \cdots \omega^{\beta_n}$ to convey that $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_k$.

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- $\beta < \varepsilon_0$ has a Cantor normal form with exponents $\beta_i < \beta$ and these exponents have Cantor normal forms with yet again smaller exponents. As this process must terminate, ordinals $< \varepsilon_0$ can be coded by natural numbers.

Coding ε_0 *in* $\mathbb{N}$

Define a function

$$\lceil\,.\,\rceil : \varepsilon_0 \longrightarrow \mathbb{N}$$

by

$$\lceil\delta\rceil = \begin{cases} 0 & \text{if } \delta = 0 \\ \langle\lceil\delta_1\rceil, \ldots, \lceil\delta_n\rceil\rangle & \text{if } \delta =_{CNF} \omega^{\delta_1} + \cdots \omega^{\delta_n} \end{cases}$$

where $\langle k_1, \cdots, k_n\rangle := 2^{k_1+1} \cdot \ldots \cdot p_n^{k_n+1}$ with p_i being the ith prime number (or any other coding of tuples). Further define

$$\begin{array}{rcl} A_0 & := & \mathbf{ran}(\lceil\,.\,\rceil) \\ \lceil\delta\rceil \prec \lceil\beta\rceil & :\Leftrightarrow & \delta < \beta \\ \lceil\delta\rceil \hat{+} \lceil\beta\rceil & := & \lceil\delta + \beta\rceil \\ \lceil\delta\rceil \hat{\cdot} \lceil\beta\rceil & := & \lceil\delta \cdot \beta\rceil \\ \hat{\omega}^{\lceil\delta\rceil} & := & \lceil\omega^{\delta}\rceil. \end{array}$$

Coding ε_0 *in* $\mathbb{N}$

Then

$$\langle \varepsilon_0, +, \cdot, \delta \mapsto \omega^\delta, < \rangle \quad \cong \quad \langle A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, \prec \rangle.$$

$A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, \prec$ are recursive, in point of fact, they are all elementary recursive.

Transfinite Induction

- **TI**$(A, \prec)$ is the schema

$$\forall n \in A\, [\forall k \prec n\, P(k) \;\to\; P(n)] \;\to\; \forall n \in A\, P(n)$$

with P arithmetical.

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- For $\alpha \in A$ let $\prec_\alpha$ be $\prec$ restricted to $A_\alpha := \{\beta \in A \mid \beta \prec \alpha\}$.

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- Every ordinal analysis of a classical or intuitionistic theory **T** that has ever appeared in the literature provides an EORS $\langle A, \lhd, \ldots \rangle$ such that **T** is finitistically reducible to

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- $|\mathbf{T}|_{\sup} = |\lhd|$.

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- **Cut Elimination** (and Proof Collapsing Techniques)
- Development of ever stronger **Ordinal Representation Systems**

- A **sequent** is an expression $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sequences of formulae $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$, respectively.

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- $\Gamma \Rightarrow \Delta$ is read, informally, as Γ yields Δ or, rather, the conjunction of the A_i yields the disjunction of the B_j.

Negation

$$\frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}\ \neg L \qquad \frac{B, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg B}\ \neg R$$

Implication

$$\frac{\Gamma \Rightarrow \Delta, A \qquad B, \Lambda \Rightarrow \Theta}{A \to B, \Gamma, \Lambda \Rightarrow \Delta, \Theta} \to L \qquad \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \to R$$

Conjunction

$$\frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \wedge L1 \qquad \frac{B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \wedge L2$$

$$\frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \wedge R$$

Disjunction

$$\frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \vee L$$

$$\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B} \vee R1 \qquad \frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \vee B} \vee R2$$

Quantifiers

$$\frac{F(t), \Gamma \Rightarrow \Delta}{\forall x\, F(x), \Gamma \Rightarrow \Delta}\ \forall L \qquad \frac{\Gamma \Rightarrow \Delta, F(a)}{\Gamma \Rightarrow \Delta, \forall x\, F(x)}\ \forall R$$

$$\frac{F(a), \Gamma \Rightarrow \Delta}{\exists x\, F(x), \Gamma \Rightarrow \Delta}\ \exists L \qquad \frac{\Gamma \Rightarrow \Delta, F(t)}{\Gamma \Rightarrow \Delta, \exists x\, F(x)}\ \exists R$$

In $\forall$L and $\exists$R, t is an arbitrary term. The variable a in $\forall$R and $\exists$L is an eigenvariable of the respective inference, i.e. a is not to occur in the lower sequent.

Identity Axiom

$$A \Rightarrow A$$

where A is any formula.

One could limit this axiom to the case of atomic formulae A

CUT

$$\frac{\Gamma \Rightarrow \Delta, A \qquad A, \Lambda \Rightarrow \Theta}{\Gamma, \Lambda \Rightarrow \Delta, \Theta}\ \text{Cut}$$

A is called the cut formula of the inference.

Example

$$\frac{B \Rightarrow A \qquad A \Rightarrow C}{B \Rightarrow C}\ \text{Cut}$$

Structural Rules

Exchange, Weakening, Contraction

$$\frac{\Gamma, A, B, \Lambda \Rightarrow \Delta}{\Gamma, B, A, \Lambda \Rightarrow \Delta}\ \mathcal{X}_l \qquad \frac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda}\ \mathcal{X}_r$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathcal{W}_l \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ \mathcal{W}_r$$

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ \mathcal{C}_l \qquad \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ \mathcal{C}_r$$

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Specifically, in the intuitionistic sequent calculus there are no inferences corresponding to contraction right or exchange right.

Classical Example

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Rightarrow A, \neg A}\ \neg\text{R}}{\Rightarrow A,\ A \vee \neg A}\ \vee\text{R}}{\Rightarrow A \vee \neg A,\ A}\ \mathcal{X}_r}{\Rightarrow A \vee \neg A,\ A \vee \neg A}\ \vee\text{R}}{\Rightarrow A \vee \neg A}\ \mathcal{C}_r$$

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Notice that the above proof is not intuitionistic since it involves sequents that are not intuitionistic.

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$$
\begin{array}{ll}
F(a) \Rightarrow F(a) & \\
\hline
F(a) \Rightarrow \exists x\, F(x) & \exists\mathrm{R} \\
\hline
\neg\exists x\, F(x), F(a) \Rightarrow & \neg\mathrm{L} \\
\hline
F(a), \neg\exists x\, F(x) \Rightarrow & \mathcal{X}_I \\
\hline
\neg\exists x F(x) \Rightarrow \neg F(a) & \neg\mathrm{L} \\
\hline
\neg\exists x\, F(x) \Rightarrow \forall x\, \neg F(x) & \forall\mathrm{R} \\
\hline
\Rightarrow \neg\exists x\, F(x) \rightarrow \forall x\, \neg F(x) & \rightarrow\mathrm{R}
\end{array}
$$

Cut Elimination

If a sequent

$$\Gamma \Rightarrow \Delta$$

is provable, then it is provable without cuts.

Cut Elimination

EXAMPLE

Here is an example of how to eliminate cuts of a special form:

$$\dfrac{\dfrac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}\ {\rightarrow}\mathrm{R} \qquad \dfrac{\Lambda \Rightarrow \Theta, A \qquad B, \Xi \Rightarrow \Phi}{A \rightarrow B, \Lambda, \Xi \Rightarrow \Theta, \Phi}\ {\rightarrow}\mathrm{L}}{\Gamma, \Lambda, \Xi \Rightarrow \Delta, \Theta, \Phi}\ \mathrm{Cut}$$

is replaced by

$$\dfrac{\dfrac{\Lambda \Rightarrow \Theta, A \qquad A, \Gamma \Rightarrow \Delta, B}{\Lambda, \Gamma \Rightarrow \Theta, \Delta, B}\ \mathrm{Cut} \qquad B, \Xi \Rightarrow \Phi}{\Gamma, \Lambda, \Xi \Rightarrow \Delta, \Theta, \Phi}\ \mathrm{Cut}$$

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Corollary *A contradiction, i.e. the empty sequent, is not deducible.*

Applications of the Haupsatz

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- Herbrand's Theorem in *LK* (classical):

$$\vdash \exists x R(x) \quad \text{implies} \quad \vdash R(t_1) \vee \ldots \vee R(t_n)$$

some t_i (R quantifier-free).

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- Hilbert-Ackermann Consistency
- If T is a **geometric theory** and T classically proves a **geometric implication** A then T intuitionistically proves A.

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- Axioms are detrimental to this procedure. It breaks down because the symmetry of the sequent calculus is lost. In general, we cannot remove cuts from deductions in a theory T when the cut formula is an axiom of T.
- However, sometimes the axioms of a theory are of bounded syntactic complexity. Then the procedure applies partially in that one can remove all cuts that exceed the complexity of the axioms of T.

Partial Cut Elimination

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- This is a very important tool in proof theory. For example, it works very well if the axioms of a theory can be presented as atomic intuitionistic sequents (also called Horn clauses), yielding the completeness of Robinsons **resolution method**.

Partial cut elimination also pays off in the case of fragments of **PA** and set theory with restricted induction schemes, be it induction on natural numbers or sets. This method can be used to extract bounds from proofs of Π^0_2 statements in such fragments.

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- The price to pay will be that deductions become infinite.

With the aid of the ω-rule, induction becomes logically deducible in infinitary logic.

Theorem For every n there is a finite deduction $\mathcal{D}_n$ of the sequent

$$F(0), \forall x\,[F(x) \rightarrow F(Sx)] \Rightarrow F(n).$$

Proof. Since $B, \Gamma \Rightarrow B$ is deducible for every formula B and sequence Γ, we obtain $\mathcal{D}_0$.

Let $\Delta := F(0), \forall x\,[F(x) \rightarrow F(Sx)]$. From $\mathcal{D}_n$ we obtain $\mathcal{D}_{n+1}$:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}_n \\ \Delta \Rightarrow F(n).\end{matrix} \qquad \begin{matrix}\mathcal{D}^* \\ F(Sn), \Delta \Rightarrow F(Sn)\end{matrix}}{F(n) \rightarrow F(Sn), \Delta \Rightarrow F(S(n))} \rightarrow \mathrm{L}}{\forall x\,[F(x) \rightarrow F(Sx)], \Delta \Rightarrow F(S(n))} \forall \mathrm{L}}{F(0), \forall x\,[F(x) \rightarrow F(Sx)] \Rightarrow F(S(n))} \mathrm{Struc}$$

Embedding Theorem

If

$$\mathbf{PA} \vdash \Gamma \Rightarrow \Delta$$

then

$$\mathbf{PA}_\omega \left|\frac{\omega+m}{k}\right. \Gamma \Rightarrow \Delta$$

for some $m, k < \omega$.

Reduction Lemma If $\mathbf{PA}_\omega \overset{\alpha}{\underset{k}{\mid\!\!-}} \Gamma \Rightarrow \Delta, A$ and $\mathbf{PA}_\omega \overset{\beta}{\underset{k}{\mid\!\!-}} A, \Lambda \Rightarrow \Theta$ with $k = |A|$, then

$$\mathbf{PA}_\omega \overset{\alpha \# \beta}{\underset{k}{\mid\!\!-}} \Gamma, \Lambda \Rightarrow \Delta, \Theta\,.$$

Cut Elimination for $\mathbf{PA}_\omega$

Theorem If $\mathbf{PA}_\omega \mid\frac{\alpha}{k+1} \Gamma \Rightarrow \Delta$, then $\mathbf{PA}_\omega \mid\frac{\omega^\alpha}{k} \Gamma \Rightarrow \Delta$.

Cut Elimination Theorem If $\mathbf{PA}_\omega \mid\frac{\alpha}{n} \Gamma \Rightarrow \Delta$, then

$$\mathbf{PA}_\omega \mid\frac{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^\alpha}}}}}}{0} \Gamma \Rightarrow \Delta \qquad \underbrace{\omega^{\omega^{\cdot^{\cdot^{\cdot^{\omega^\alpha}}}}}}_{n \text{ times}}.$$

To achieve (partial) cut elimination for set theory, one needs infinitary rules similar to the ω-rule. These rules enable one to get cut-free deductions of $\in$-induction.

$$\forall x\ [[\forall y \in x\, A(y)] \rightarrow A(x)]\ \rightarrow\ \forall x\, A(x)$$

β-Logic

A brief history of ordinal representation systems

1904-1950

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Hardy gives explicit representations for all ordinals $< \omega^2$.

O. Veblen, 1908

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- He applied two new operations to **continuous increasing functions** on ordinals:

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Veblen extended the initial segment of the countable for which fundamental sequences can be given effectively.

- He applied two new operations to **continuous increasing functions** on ordinals:
  - **Derivation**
  - **Transfinite Iteration**
- Let **ON** be the class of ordinals. A (class) function f : **ON** $\to$ **ON** is said to be **increasing** if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$ and **continuous** (in the order topology on **ON**) if

$$f(\lim_{\xi<\lambda} \alpha_\xi) = \lim_{\xi<\lambda} f(\alpha_\xi)$$

holds for every limit ordinal λ and increasing sequence $(\alpha_\xi)_{\xi<\lambda}$.

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- The **derivative** f' of a function f : **ON** $\rightarrow$ **ON** is the function which enumerates in increasing order the solutions of the equation

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- The **derivative** f' of a function $f : \mathbf{ON} \to \mathbf{ON}$ is the function which enumerates in increasing order the solutions of the equation

$$f(\alpha) = \alpha,$$

  also called the **fixed points** of f.
- If f is a normal function,

$$\{\alpha : f(\alpha) = \alpha\}$$

  is a proper class and f' will be a normal function, too.

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- $f_0 = f$
- $f_{\alpha+1} = f_\alpha{}'$
- $$f_\lambda(\xi) = \xi^{th} \text{ element of } \bigcap_{\alpha<\lambda} \{\text{Fixed points of } f_\alpha\} \qquad \text{for } \lambda \text{ limit.}$$

The Feferman-Schütte Ordinal Γ_0

- From the normal function f we get a two-place function,

$$\varphi_f(\alpha, \beta) := f_\alpha(\beta).$$

Veblen then discusses the hierarchy when

$$f = \ell, \qquad \ell(\alpha) = \omega^\alpha.$$

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- The least ordinal $\gamma > 0$ closed under φ_ℓ, i.e. the least ordinal > 0 satisfying

$$(\forall \alpha, \beta < \gamma)\ \varphi_\ell(\alpha, \beta) < \gamma$$

is the famous ordinal Γ_0 which Feferman and Schütte determined to be the least ordinal 'unreachable' by **predicative means**.

The Big Veblen Number

- Veblen extended this idea first to arbitrary **finite numbers of arguments**, but then also to **transfinite numbers of arguments**, with the proviso that in, for example

$$\Phi_f(\alpha_0, \alpha_1, \ldots, \alpha_\eta),$$

only a finite number of the arguments

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may be non-zero.

- Veblen singled out the ordinal $E(0)$, where $E(0)$ is the least ordinal $\delta > 0$ which cannot be named in terms of functions

$$\Phi_\ell(\alpha_0, \alpha_1, \ldots, \alpha_\eta)$$

with $\eta < \delta$, and each $\alpha_\gamma < \delta$.

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- Define a set of ordinals $\mathfrak{B}$ closed under successor such that with each limit $\lambda \in \mathfrak{B}$ is associated an increasing sequence $\langle \lambda[\xi] : \xi < \tau_\lambda \rangle$ of ordinals $\lambda[\xi] \in \mathfrak{B}$ of length $\tau_\lambda \leq \mathfrak{B}$ and $\lim_{\xi < \tau_\lambda} \lambda[\xi] = \lambda$.

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- Let Ω be the first uncountable ordinal. A hierarchy of functions $(\varphi^{\mathfrak{B}}_\alpha)_{\alpha \in \mathfrak{B}}$ is then obtained as follows:

$$\varphi^{\mathfrak{B}}_0(\beta) = 1 + \beta \qquad \varphi^{\mathfrak{B}}_{\alpha+1} = \left(\varphi^{\mathfrak{B}}_\alpha\right)'$$

$$\varphi^{\mathfrak{B}}_\lambda \text{ enumerates } \bigcap_{\xi < \tau_\lambda} (\text{Range of } \varphi^{\mathfrak{B}}_{\lambda[\xi]}) \qquad \lambda \text{ limit, } \tau_\lambda < \Omega$$

$$\varphi^{\mathfrak{B}}_\lambda \text{ enumerates } \{\beta < \Omega : \varphi^{\mathfrak{B}}_{\lambda[\beta]}(0) = \beta\} \qquad \lambda \text{ limit, } \tau_\lambda = \Omega.$$

1960-1974

After Bachmann, the story of ordinal representation systems becomes very complicated.

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- **Bridge** and **Buchholz** showed computability of systems obtained by Feferman's approach.

“Natural” well-orderings

Set-theoretical (*Cantor, Veblen, Gentzen, Bachmann, Schütte, Feferman, Pfeiffer, Isles, Bridge, Buchholz, Pohlers, Jäger, Rathjen*)

- Define hierarchies of functions on the ordinals.
- Build up terms from function symbols for those functions.
- The ordering on the values of terms induces an ordering on the terms.

Reductions in proof figures (*Takeuti, Yasugi, Kino, Arai*)

- Ordinal diagrams; formal terms endowed with an inductively defined ordering on them.

"Natural" well-orderings

Patterns of elementary substructurehood (*Carlson*)

- Finite structures with Σ_n-elementary substructure relations .

Category-theoretical (*Aczel, Girard, Jervell, Vauzeilles*)

- Functors on the category of ordinals (with strictly increasing functions) respecting direct limits and pull-backs.

Representation systems from below (*Setzer*)

Second order arithmetic; $\mathbf{Z_2}$ *aka Analysis*

$\mathbf{Z_2}$ is a two sorted formal system. Extends **PA**.

- Variables $n, m, \ldots$ range over natural numbers.
  Variables $X, Y, Z, \ldots$ range over sets of natural numbers.
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- Comprehension Principle/Axiom:

  For any property P definable in the language of $\mathbf{Z_2}$,

$$\{n \in \mathbb{N} \mid P(n)\}$$

  is a set; or more formally

$$\text{(CA)} \qquad \exists X \, \forall n \, [n \in X \leftrightarrow A(x)]$$

  for any formula $A(x)$ of $\mathbf{Z_2}$.

Stratification of Comprehension

- A Π^1_k-formula (Σ^1_k-formula) is a formula of $\mathbf{Z_2}$ of the form

  $$\forall X_1 \ldots QX_k\, A(X_1, \ldots, X_k) \qquad (\exists X_1 \ldots QX_k\, A(X_1, \ldots, X_k))$$

  with $\forall X_1 \ldots QX_k$ ($\exists X_1 \ldots QX_k$) a string of k alternating set quantifiers, beginning with a universal quantifier (existential quantifier), followed by a formula $A(X_1, \ldots, X_k)$ without set quantifiers.

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- Π^1_k-comprehension (Σ^1_k-comprehension) is the scheme

  $$\exists X\, \forall n\, [n \in X \leftrightarrow A(x)]$$

  with $A(x)$ Π^1_k (Σ^1_k).

Subsystems of $\mathbf{Z_2}$

- Basic arithmetical axioms in all subtheories of $\mathbf{Z_2}$ are: defining axioms for $0, 1, +, \times, E, <$ (as for **PA**) and the *induction axiom*

$$\forall X\,[0 \in X \wedge \forall n(n \in X \to n+1 \in X) \to \forall n\,(n \in X)].$$

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- $(\mathbf{Ax})$ stands for the theory $(\mathbf{Ax})_0$ augmented by the scheme of induction for all $\mathcal{L}_2$-formulae.
- Let $\mathcal{F}$ be a collection of formulae of $\mathbf{Z_2}$. Another important axiom scheme for formulae F in $\mathcal{C}$ is

$$\mathcal{C}-\mathbf{AC} \qquad \forall n \exists Y F(n, Y) \rightarrow \exists Y \forall n F(x, Y_n),$$

where $Y_n := \{m : 2^n 3^m \in Y\}$.

How much of $\mathbf{Z_2}$ *is needed?*

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- Minimal foundational frameworks for Ordinary Mathematics:
  Feferman, **Lorenzen**, **Takeuti** ....
- **Reverse Mathematics**, early 1970s-now
  H. Friedman, **S. Simpson**, ...

Given a specific theorem τ of ordinary mathematics, which set existence axioms are needed in order to prove τ?

Five Systems

For many mathematical theorems τ, there is a weakest natural subsystem $S(\tau)$ of $\mathbf{Z_2}$ such that $S(\tau)$ proves τ.
Moreover, it has turned out that $S(\tau)$ often belongs to a small list of specific subsystems of $\mathbf{Z_2}$. Reverse Mathematics has singled out five subsystems of $\mathbf{Z_2}$:

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- $(\Pi^1_1-\mathbf{CA})_0$ Π^1_1-Comprehension

Mathematical Equivalences: Examples

- **$\mathbf{RCA_0}$** “Every countable field has an algebraic closure"; “Every countable ordered field has a real closure"

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- **$\mathbf{ATR}_0$** “Every countable reduced abelian *p*-group has an Ulm resolution"
- $(\Pi^1_1-\mathbf{CA})_0$ “Every uncountable closed set of real numbers is the union of a perfect set and a countable set"; “Every countable abelian group is a direct sum of a divisible group and a reduced group"

$$|\mathbf{ATR}_0| = \Gamma_0$$

$$|\mathbf{ACA}_0| = \varepsilon_0$$

$$|\mathbf{RCA}_0| = \omega^\omega = |\mathbf{WKL}_0|$$

$$0$$

$$|(\Sigma^1_2\text{-}\mathbf{AC}) + \mathbf{BI}| = \psi_{\Omega_1} I$$

$$|(\Delta^1_2\text{-}\mathbf{CA})| = \psi_{\Omega_1}\Omega_{\varepsilon_0}$$

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- Takeuti, Yasugi 1983
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A Brief History of Ordinal Analysis cont'd

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  Constructible Hierarchy in Proof Theory

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- Jäger, Pohlers 1982
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- Arai Ordinal Analysis of Theories up to Π^1_2-Comprehension using Reductions on Finite Proof Figures and Ordinal Diagrams.