PROOF THEORY:
From arithmetic to set theory

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Plan of First and Second Talk

- The origins of Proof theory: Hilbert’s Programme
- Gentzen’s Result
- The General Form of Ordinal Analysis
- Gentzen’s Hauptsatz: Cut Elimination
- A Brief History of Ordinal Representation Systems
- A Brief History of Ordinal Analyses
- Applications of Ordinal Analysis
  1. Combinatorial Independence Results
  2. Classification of Provable Functions
  3. Equiconsistency Results
Plan of the Third and Fourth Talk

PREDICATIVE PROOF THEORY

IMPREDICATIVE PROOF THEORY

- Ordinal Analysis of Kripke-Platek Set Theory (sketch)
- Uniformity of Infinite Proofs
- Proof Theory of Much Stronger Theories
The Origins of Proof Theory (Beweistheorie)

- Hilbert’s second problem (1900): Consistency of Analysis
- Hilbert’s Programme (1922, 1925)
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To carry out this task, Hilbert inaugurated a new mathematical discipline: Beweistheorie (Proof Theory).

In Hilbert’s Proof Theory, proofs become mathematical objects sui generis.
Consistency proof for a second-order version of Primitive Recursive Arithmetic.

Uses a finitistic version of transfinite induction up to the ordinal $\omega^\omega$. 
• **Gerhard Gentzen** showed that transfinite induction up to the ordinal

\[ \varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \ldots\} = \text{least } \alpha. \omega^\alpha = \alpha \]

suffices to prove the **consistency** of Peano Arithmetic, PA.
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$$

suffices to prove the consistency of Peano Arithmetic, PA.

Gentzen’s applied transfinite induction up to \(\varepsilon_0\) solely to primitive recursive predicates and besides that his proof used only finitistically justified means.
Gentzen’s Result in Detail

\[ F + \text{PR-TI}(\varepsilon_0) \vdash \text{Con}(\text{PA}), \]

where \( F \) signifies a theory that is acceptable in finitism (e.g. \( F = \text{PRA} = \text{Primitive Recursive Arithmetic} \)) and \( \text{PR-TI}(\varepsilon_0) \) stands for transfinite induction up to \( \varepsilon_0 \) for primitive recursive predicates.
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- Gentzen also showed that his result is best possible: \( \text{PA} \) proves transfinite induction up to \( \alpha \) for arithmetic predicates for any \( \alpha < \varepsilon_0 \).
The non-finitist part of PA is encapsulated in PR-TI(\(\varepsilon_0\)) and therefore “measured” by \(\varepsilon_0\), thereby tempting one to adopt the following definition of proof-theoretic ordinal of a theory \(T\):

\[
|T|_{\text{Con}} = \text{least } \alpha. \text{ PRA } + \text{PR-TI}(\alpha) \vdash \text{Con}(T).
\]
The supremum of the provable ordinals

• \( \langle A, \prec \rangle \) is said to be provably wellordered in \( T \) if

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- $\langle A, \prec \rangle$ is said to be provably wellordered in $T$ if
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- $\alpha$ is provably computable in $T$ if there is a computable well-ordering $\langle A, \prec \rangle$ with order-type $\alpha$ such that
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- The supremum of the provable well-orderings of \( T \):
  \[ |T|_{\text{sup}} := \sup \{ \alpha : \alpha \text{ provably computable in } T \}. \]
Ordinal Structures

We are interested in representing specific ordinals $\alpha$ as relations on $\mathbb{N}$.

Natural ordinal representation systems are frequently derived from structures of the form

$$\mathfrak{A} = \langle \alpha, f_1, \ldots, f_n, <_{\alpha} \rangle$$

where $\alpha$ is an ordinal, $<_{\alpha}$ is the ordering of ordinals restricted to elements of $\alpha$ and the $f_i$ are functions

$$f_i : \underbrace{\alpha \times \cdots \times \alpha}_{k_i \text{ times}} \rightarrow \alpha$$

for some natural number $k_i$. 
Ordinal Representation Systems

\[ \mathbb{A} = \langle A, g_1, \ldots, g_n, \prec \rangle \]

is a **computable** (or **recursive**) representation of
\[ \mathbb{A} = \langle \alpha, f_1, \ldots, f_n, \prec_\alpha \rangle \]
if the following conditions hold:

1. \( A \subseteq \mathbb{N} \) and \( A \) is a computable set.
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2. \( \prec \) is a computable total ordering on \( A \) and the functions \( g_i \) are computable.
3. \( \mathcal{A} \cong \mathcal{A} \), i.e. the two structures are isomorphic.
Theorem (Cantor, 1897) For every ordinal $\beta > 0$ there exist unique ordinals $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n$ such that

$$\beta = \omega^{\beta_0} + \cdots + \omega^{\beta_n}. \quad (1)$$

The representation of $\beta$ in (1) is called the Cantor normal form.

We shall write $\beta =_{\text{CNF}} \omega^{\beta_1} + \cdots \omega^{\beta_n}$ to convey that $\beta_0 \geq \beta_1 \geq \cdots \geq \beta_k$. 

FROM ARITHMETIC TO SET THEORY
A Representation for $\varepsilon_0$

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From arithmetic to set theory
A Representation for \( \varepsilon_0 \)

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  \]

- \( \varepsilon_0 \) is the least ordinal \( \alpha \) such that \( \omega^\alpha = \alpha \).

- \( \beta < \varepsilon_0 \) has a Cantor normal form with exponents \( \beta_i < \beta \) and these exponents have Cantor normal forms with yet again smaller exponents. As this process must terminate, ordinals \(< \varepsilon_0 \) can be coded by natural numbers.
Define a function
\[ [\ . ] : \varepsilon_0 \rightarrow \mathbb{N} \]
by
\[
[\delta] = \begin{cases}
0 & \text{if } \delta = 0 \\
\langle [\delta_1], \ldots, [\delta_n] \rangle & \text{if } \delta = \text{CNF } \omega^{\delta_1} + \cdots + \omega^{\delta_n}
\end{cases}
\]
where \( \langle k_1, \ldots, k_n \rangle := 2^{k_1+1} \cdot \ldots \cdot p_n^{k_n+1} \) with \( p_i \) being the \( i \)th prime number (or any other coding of tuples). Further define
\[
A_0 := \text{ran}( [\ . ] )
\]
\[
[\delta] < [\beta] :\Leftrightarrow \delta < \beta
\]
\[
[\delta] + [\beta] := [\delta + \beta]
\]
\[
[\delta] \cdot [\beta] := [\delta \cdot \beta]
\]
\[
\hat{\omega}^\delta := [\omega^\delta].
\]
Coding $\varepsilon_0$ in $\mathbb{N}$

Then

$$\langle \varepsilon_0, +, \cdot, \delta \mapsto \omega^\delta, < \rangle \cong \langle A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, < \rangle.$$  

$A_0, \hat{+}, \hat{\cdot}, x \mapsto \hat{\omega}^x, <$ are recursive, in point of fact, they are all elementary recursive.
Transfinite Induction

- \(\text{TI}(A, \prec)\) is the schema

\[
\forall n \in A \left[ \forall k \prec n P(k) \rightarrow P(n) \right] \rightarrow \forall n \in A P(n)
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with \(P\) arithmetical.
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- For \(\alpha \in A\) let \(\prec_\alpha\) be \(\prec\) restricted to \(A_\alpha := \{\beta \in A \mid \beta \prec \alpha\}\).
The general form of ordinal analysis

- $T$ framework for formalizing a certain part of mathematics. $T$ should be a true theory which contains a modicum of arithmetic.

- Every ordinal analysis of a classical or intuitionistic theory $T$ that has ever appeared in the literature provides an $EORS \langle A, \alpha, \ldots \rangle$ such that $T$ is finitistically reducible to $PA + \bigcup_{\alpha \in A} TI(A, \alpha)$.

- $T$ and $HA + \bigcup_{\alpha \in A} TI(A, \alpha)$ prove the same $\Pi^0_2$ sentences.

- $|T|_{sup} = |T|$. 

FROM ARITHMETIC TO SET THEORY
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- $|T|_{\text{sup}} = |\prec|$. 
The two main strands of research are:

- **Cut Elimination** (and **Proof Collapsing Techniques**)
Ordinarily Informative Proof Theory

The two main strands of research are:

- **Cut Elimination** (and Proof Collapsing Techniques)
- Development of ever stronger **Ordinal Representation Systems**
A **sequent** is an expression \( \Gamma \Rightarrow \Delta \) where \( \Gamma \) and \( \Delta \) are finite sequences of formulae \( A_1, \ldots, A_n \) and \( B_1, \ldots, B_m \), respectively.
A **sequent** is an expression $\Gamma \Rightarrow \Delta$ where $\Gamma$ and $\Delta$ are finite sequences of formulae $A_1, \ldots, A_n$ and $B_1, \ldots, B_m$, respectively.

$\Gamma \Rightarrow \Delta$ is read, informally, as $\Gamma$ yields $\Delta$ or, rather, the conjunction of the $A_i$ yields the disjunction of the $B_j$. 
Negation

\[ \Gamma \Rightarrow \Delta, A \]
\[ \neg A, \Gamma \Rightarrow \Delta \]
\[ \neg L \]

\[ B, \Gamma \Rightarrow \Delta \]
\[ \Gamma \Rightarrow \Delta, \neg B \]
\[ \neg R \]

Implication

\[ \Gamma \Rightarrow \Delta, A \]
\[ B, \Lambda \Rightarrow \Theta \]
\[ A \rightarrow B, \Gamma, \Lambda \Rightarrow \Delta, \Theta \]
\[ \rightarrow L \]

\[ A, \Gamma \Rightarrow \Delta, B \]
\[ \Gamma \Rightarrow \Delta, A \rightarrow B \]
\[ \rightarrow R \]
Conjunction

\[
\frac{A, \Gamma \Rightarrow \Delta}{A \land B, \Gamma \Rightarrow \Delta} \land L1
\]

\[
\frac{B, \Gamma \Rightarrow \Delta}{A \land B, \Gamma \Rightarrow \Delta} \land L2
\]

\[
\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \land B} \land R
\]

Disjunction

\[
\frac{A, \Gamma \Rightarrow \Delta}{A \lor B, \Gamma \Rightarrow \Delta} \lor L
\]

\[
\frac{B, \Gamma \Rightarrow \Delta}{A \lor B, \Gamma \Rightarrow \Delta} \lor L
\]

\[
\frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \lor B} \lor R1
\]

\[
\frac{\Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \lor B} \lor R2
\]

FROM ARITHMETIC TO SET THEORY
The Sequent Calculus

LOGICAL INFERENCES II

Quantifiers

\[
\begin{align*}
F(t), \Gamma \Rightarrow \Delta & \Rightarrow \forall L, \forall x F(x), \Gamma \Rightarrow \Delta \\
F(a), \Gamma \Rightarrow \Delta & \Rightarrow \exists L, \exists x F(x), \Gamma \Rightarrow \Delta
\end{align*}
\]

\[
\begin{align*}
\Gamma \Rightarrow \Delta, F(a) & \Rightarrow \forall R, \forall x F(x) \\
\Gamma \Rightarrow \Delta, F(t) & \Rightarrow \exists R, \exists x F(x)
\end{align*}
\]

In \(\forall L\) and \(\exists R\), \(t\) is an arbitrary term. The variable \(a\) in \(\forall R\) and \(\exists L\) is an eigenvariable of the respective inference, i.e. \(a\) is not to occur in the lower sequent.
Identity Axiom

$$A \Rightarrow A$$

where $A$ is any formula.

One could limit this axiom to the case of atomic formulae $A$. 
\[ \text{CUT} \]

\[
\frac{\Gamma \Rightarrow \Delta, A \quad A, \Lambda \Rightarrow \Theta}{\Gamma, \Lambda \Rightarrow \Delta, \Theta} \quad \text{Cut}
\]

\( A \) is called the \textit{cut formula} of the inference.

\textbf{Example}

\[
\frac{B \Rightarrow A \quad A \Rightarrow C}{B \Rightarrow C} \quad \text{Cut}
\]
### Structural Rules

- **Exchange ($\chi_l$)**
  
  \[ \Gamma, A, B, \Lambda \Rightarrow \Delta \quad \Rightarrow \quad \Gamma, B, A, \Lambda \Rightarrow \Delta \]

- **Weakening ($\mathcal{W}_l$)**
  
  \[ \Gamma \Rightarrow \Delta \quad \Rightarrow \quad \Gamma, A \Rightarrow \Delta \]

- **Contraction ($C_l$)**
  
  \[ \Gamma, A, A \Rightarrow \Delta \quad \Rightarrow \quad \Gamma, A \Rightarrow \Delta \]

### Exchange, Weakening, Contraction

- **Exchange ($\chi_r$)**
  
  \[ \Gamma \Rightarrow \Delta, A, B, \Lambda \quad \Rightarrow \quad \Gamma \Rightarrow \Delta, B, A, \Lambda \]

- **Weakening ($\mathcal{W}_r$)**
  
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- **Contraction ($C_r$)**
  
  \[ \Gamma \Rightarrow \Delta, A, A \quad \Rightarrow \quad \Gamma \Rightarrow \Delta, A \]
The INTUITIONISTIC case

The intuitionistic sequent calculus is obtained by requiring that all sequents be intuitionistic. A sequent \( \Gamma \Rightarrow \Delta \) is said to be intuitionistic if \( \Delta \) consists of at most one formula. Specifically, in the intuitionistic sequent calculus there are no inferences corresponding to contraction right or exchange right.
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Classical Example

Our first example is a deduction of the law of excluded middle.

\[ A \Rightarrow A \]

\[ \neg R \Rightarrow A \]

\[ A \lor \neg A \]

Notice that the above proof is not intuitionistic since it involves sequents that are not intuitionistic.
Our first example is a deduction of the law of excluded middle.
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\[
\frac{A \Rightarrow A}{\Rightarrow A, \neg A} \quad \neg R
\]
\[
\Rightarrow A, A \lor \neg A \quad \lor R
\]
\[
\Rightarrow A \lor \neg A, A \quad \lor L
\]
\[
\Rightarrow A \lor \neg A \quad \lor R
\]

\[
\Rightarrow A \lor \neg A
\]
Our first example is a deduction of the law of excluded middle.

\[
\begin{align*}
A & \Rightarrow A \\
\Rightarrow A, \neg A & \quad \neg \text{R}
\end{align*}
\]

\[
\begin{align*}
\Rightarrow A, A \lor \neg A & \quad \lor \text{R} \\
\Rightarrow A \lor \neg A, A & \quad \lor \text{R} \\
\Rightarrow A \lor \neg A, A \lor \neg A & \quad \lor \text{R} \\
\Rightarrow A \lor \neg A & \quad C_r
\end{align*}
\]

Notice that the above proof is not intuitionistic since it involves sequents that are not intuitionistic.
Intuitionistic Example

The second example is an intuitionistic deduction.

\[ F(a) \Rightarrow F(a) \]

\[ \exists R F(a) \Rightarrow \exists x F(x) \]

\[ \neg L \neg \exists x F(x) , F(a) \Rightarrow X \]

\[ \neg F(a) \Rightarrow \neg \exists x F(x) \Rightarrow \neg \forall x \neg F(x) \]

\[ \rightarrow R \Rightarrow \neg \exists x F(x) \Rightarrow \forall x \neg F(x) \]
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\[
\begin{array}{c}
F(a) \Rightarrow F(a) \\
F(a) \Rightarrow \exists x F(x) \\
\neg \exists x F(x), F(a) \Rightarrow \\
\neg \exists x F(x) \Rightarrow \neg F(a) \\
\neg \exists x F(x) \Rightarrow \forall x \neg F(x) \\
\Rightarrow \neg \exists x F(x) \rightarrow \forall x \neg F(x)
\end{array}
\]
Cut Elimination

If a sequent

\[ \Gamma \Rightarrow \Delta \]

is provable, then it is provable \textit{without cuts}. 
Here is an example of how to eliminate cuts of a special form:

\[
\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \quad \text{R} \quad \frac{\Lambda \Rightarrow \Theta, A}{A \rightarrow B, \Lambda, \Xi \Rightarrow \Theta, \Phi} \quad \text{L}
\]

is replaced by

\[
\frac{\Lambda \Rightarrow \Theta, A}{A, \Gamma \Rightarrow \Delta, B} \quad \text{Cut} \quad \frac{B, \Xi \Rightarrow \Phi}{\Lambda, \Gamma \Rightarrow \Theta, \Delta, B} \quad \text{Cut}
\]

\[
\frac{\Gamma \Rightarrow \Delta, A \rightarrow B}{\Gamma, \Lambda, \Xi \Rightarrow \Delta, \Theta, \Phi} \quad \text{R} \quad \frac{A \rightarrow B, \Lambda, \Xi \Rightarrow \Theta, \Phi}{\Gamma, \Lambda, \Xi \Rightarrow \Delta, \Theta, \Phi} \quad \text{L}
\]
The Subformula Property

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The **Subformula Property**

*If a sequent* \( \Gamma \Rightarrow \Delta \) *is provable, then it has a deduction all of whose formulae are subformulae of the formulae in* \( \Gamma \) *and* \( \Delta \).
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The **Subformula Property**

If a sequent $\Gamma \Rightarrow \Delta$ is provable, then it has a deduction all of whose formulae are subformulae of the formulae in $\Gamma$ and $\Delta$.

**Corollary** A contradiction, i.e. the empty sequent, is not deducible.
Applications of the Haupsatz

- Herbrand's Theorem in LK (classical):
  \[ \vdash \exists x R(x) \implies \vdash R(t_1) \lor \ldots \lor R(t_n) \]
  some \( t_i \) (\( R \) quantifier-free).

- Extended Herbrand's Theorem in LK:
  \[ \vdash \Gamma \Rightarrow \exists x R(x) \implies \vdash \Gamma \Rightarrow R(t_1) \lor \ldots \lor R(t_n) \]
  some \( t_i \) (\( R \) quantifier-free, \( \Gamma \) purely universal).

- In LJ (intuitionistic):
  \[ \vdash \exists x R(x) \implies \vdash R(t) \]
  for some term \( t \).

- Hilbert-Ackermann Consistency

- If \( T \) is a geometric theory and \( T \) classically proves a geometric implication \( A \) then \( T \) intuitionistically proves \( A \).
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FROM ARITHMETIC TO SET THEORY
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- **Hilbert-Ackermann Consistency**
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  for some term $t$.

- Hilbert-Ackermann Consistency

- If $T$ is a geometric theory and $T$ classically proves a geometric implication $A$ then $T$ intuitionistically proves $A$. 
Theories and Cut Elimination

- What happens when we try to apply the procedure of cut elimination to theories?

Axioms are detrimental to this procedure. It breaks down because the symmetry of the sequent calculus is lost. In general, we cannot remove cuts from deductions in a theory \( T \) when the cut formula is an axiom of \( T \).

However, sometimes the axioms of a theory are of bounded syntactic complexity. Then the procedure applies partially in that one can remove all cuts that exceed the complexity of the axioms of \( T \).
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ROM ARITHMETIC TO SET THEORY
Theories and Cut Elimination

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Partial Cut Elimination

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Partial Cut Elimination

- Gives rise to partial cut elimination.
- This is a very important tool in proof theory. For example, it works very well if the axioms of a theory can be presented as atomic intuitionistic sequents (also called Horn clauses), yielding the completeness of Robinsons resolution method.
Partial cut elimination also pays off in the case of fragments of PA and set theory with restricted induction schemes, be it induction on natural numbers or sets. This method can be used to extract bounds from proofs of $\Pi^0_2$ statements in such fragments.
Going Infinite

- Full arithmetic, i.e. PA, does not even allow for partial cut elimination since the induction axioms have unbounded complexity.

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\[
\begin{align*}
\Gamma & \Rightarrow \Delta, F(0); \\
\Gamma & \Rightarrow \Delta, F(1); \\
& \ldots \\
\Gamma & \Rightarrow \Delta, F(n); \\
& \ldots \\
\Gamma & \Rightarrow \Delta, \forall x F(x) \\
\end{align*}
\]

\( \omega R \)

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\Gamma &\Rightarrow \Delta, F(0); \Gamma \Rightarrow \Delta, F(1); \ldots ; \Gamma \Rightarrow \Delta, F(n); \ldots \\
\Gamma &\Rightarrow \Delta, \forall x F(x) \quad \omega R
\end{align*}
\]

\[
\begin{align*}
F(0), \Gamma &\Rightarrow \Delta; F(1), \Gamma \Rightarrow \Delta; \ldots ; F(n), \Gamma \Rightarrow \Delta; \ldots \\
\exists x F(x), \Gamma &\Rightarrow \Delta \quad \omega L
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- The price to pay will be that deductions become infinite.
With the aid of the $\omega$-rule, induction becomes logically
deducible in infinitary logic.

**Theorem** For every $n$ there is a finite deduction $\mathcal{D}_n$ of the
sequent

$$F(0), \forall x [F(x) \rightarrow F(Sx)] \Rightarrow F(n).$$

**Proof.** Since $B, \Gamma \Rightarrow B$ is deducible for every formula $B$ and
sequence $\Gamma$, we obtain $\mathcal{D}_0$.

Let $\Delta := F(0), \forall x [F(x) \rightarrow F(Sx)]$. From $\mathcal{D}_n$ we obtain $\mathcal{D}_{n+1}$:

\[
\begin{align*}
\mathcal{D}_n \\
\Delta \Rightarrow F(n). \\
F(Sn), \Delta \Rightarrow F(Sn)
\end{align*}
\]

\[
\begin{align*}
&D_n \\
&\overset{\Delta \Rightarrow F(n)}{\Delta \Rightarrow F(n) \rightarrow F(Sn)} \\
&\overset{F(Sn), \Delta \Rightarrow F(Sn)}{\forall x [F(x) \rightarrow F(Sx)], \Delta \Rightarrow F(S(n))}
\end{align*}
\]

\[
\begin{align*}
&F(0), \forall x [F(x) \rightarrow F(Sx)] \Rightarrow F(S(n)) \\
&\text{Struct}
\end{align*}
\]
Embedding Theorem

If

\[ \text{PA} \vdash \Gamma \Rightarrow \Delta \]

then

\[ \text{PA}_\omega \vdash_{\text{PA}_\omega} \Gamma \Rightarrow \Delta \]

for some \( m, k < \omega \).
Reduction Lemma

If $\text{PA}_\omega \vdash^k \Gamma \Rightarrow \Delta, \mathcal{A}$ and $\text{PA}_\omega \vdash^k \mathcal{A}, \Lambda \Rightarrow \Theta$ with $k = |\mathcal{A}|$, then

$\text{PA}_\omega \vdash^k \Gamma, \Lambda \Rightarrow \Delta, \Theta$. 
Cut Elimination for $\text{PA}_\omega$

**Theorem** If $\text{PA}_\omega \vdash_{k+1}^{\alpha} \Gamma \Rightarrow \Delta$, then $\text{PA}_\omega \vdash_{k}^{\omega^\alpha} \Gamma \Rightarrow \Delta$.

**Cut Elimination Theorem** If $\text{PA}_\omega \vdash_{n}^{\alpha} \Gamma \Rightarrow \Delta$, then

$$\text{PA}_\omega \vdash^{\omega^{\omega^\ldots^{\omega^\alpha}}}_{0} \Gamma \Rightarrow \Delta$$

$n$ times
To achieve (partial) cut elimination for set theory, one needs infinitary rules similar to the $\omega$-rule. These rules enable one to get cut-free deductions of $\in$-induction.

$$\forall x \ [\forall y \in x A(y) \to A(x)] \to \forall x A(x)$$

$\beta$-Logic
A brief history of ordinal representation systems
1904-1950
Hardy (1904) wanted to “construct” a subset of $\mathbb{R}$ of size $\aleph_1$. 
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Hardy gives explicit representations for all ordinals $< \omega^2$. 
Veblen extended the initial segment of the countable for which fundamental sequences can be given effectively.

- He applied two new operations to continuous increasing functions on ordinals:

\[ f : \text{ON} \to \text{ON} \] is said to be increasing if \( \alpha < \beta \) implies \( f(\alpha) < f(\beta) \) and continuous (in the order topology on \( \text{ON} \)) if

\[ f(\lim_{\xi < \lambda} \alpha_{\xi}) = \lim_{\xi < \lambda} f(\alpha_{\xi}) \]

holds for every limit ordinal \( \lambda \) and increasing sequence \( (\alpha_{\xi})_{\xi < \lambda} \).
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O. Veblen, 1908

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- He applied two new operations to **continuous increasing functions** on ordinals:
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- Let $\text{ON}$ be the class of ordinals. A (class) function $f : \text{ON} \to \text{ON}$ is said to be **increasing** if $\alpha < \beta$ implies $f(\alpha) < f(\beta)$ and **continuous** (in the order topology on $\text{ON}$) if
  \[ f(\lim_{\xi<\lambda} \alpha_\xi) = \lim_{\xi<\lambda} f(\alpha_\xi) \]

  holds for every limit ordinal $\lambda$ and increasing sequence $(\alpha_\xi)_{\xi<\lambda}$. 
derivations

- $f$ is called **normal** if it is increasing and continuous.

- The function $\beta \mapsto \omega + \beta$ is normal while $\beta \mapsto \beta + \omega$ is not continuous at $\omega$ since $\lim_{\xi < \omega} (\xi + \omega) = \omega$ but $(\lim_{\xi < \omega} \xi) + \omega = \omega + \omega$.

- The derivative $f'$ of a function $f: \mathbb{O} \rightarrow \mathbb{O}$ is the function which enumerates in increasing order the solutions of the equation $f(\alpha) = \alpha$, also called the **fixed points** of $f$.

- If $f$ is a normal function, $\{\alpha : f(\alpha) = \alpha\}$ is a proper class and $f'$ will be a normal function, too.
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  - \( f_{\alpha + 1} = f_{\alpha}^{'} \)
A Hierarchy of Ordinal Functions

Given a normal function $f : \text{ON} \to \text{ON}$, define a hierarchy of normal functions as follows:

- $f_0 = f$
- $f_{\alpha+1} = f_\alpha'$
- $f_\lambda(\xi) = \xi^{th}$ element of $\bigcap_{\alpha < \lambda} \{\text{Fixed points of } f_\alpha\}$ for $\lambda$ limit.
The Feferman-Schütte Ordinal \( \Gamma_0 \)

- From the normal function \( f \) we get a two-place function,
  \[
  \varphi_f(\alpha, \beta) := f_\alpha(\beta).
  \]
  Veblen then discusses the hierarchy when
  \[
  f = \ell, \quad \ell(\alpha) = \omega^\alpha.
  \]
The Feferman-Schütte Ordinal $\Gamma_0$

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  \[ \varphi_f(\alpha, \beta) := f_\alpha(\beta). \]

Veblen then discusses the hierarchy when
\[ f = \ell, \quad \ell(\alpha) = \omega^\alpha. \]

- The least ordinal $\gamma > 0$ closed under $\varphi_\ell$, i.e. the least ordinal $> 0$ satisfying
  \[ (\forall \alpha, \beta < \gamma) \varphi_\ell(\alpha, \beta) < \gamma \]
  is the famous ordinal $\Gamma_0$ which Feferman and Schütte determined to be the least ordinal ‘unreachable’ by predicative means.
Veblen extended this idea first to arbitrary **finite numbers of arguments**, but then also to **transfinite numbers of arguments**, with the proviso that in, for example

\[ \Phi_f(\alpha_0, \alpha_1, \ldots, \alpha_\eta), \]

only a finite number of the arguments

\[ \alpha_\nu \]

can be non-zero.

\[ \]
The Big Veblen Number

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may be non-zero.

• Veblen singled out the ordinal **E(0)**, where **E(0)** is the least ordinal \( \delta > 0 \) which cannot be named in terms of functions

\[ \Phi_\ell(\alpha_0, \alpha_1, \ldots, \alpha_\eta) \]

with \( \eta < \delta \), and each \( \alpha_\gamma < \delta \).
Bachmann’s novel idea: Use **uncountable ordinals** to keep track of the functions defined by **diagonalization**.

Let $\Omega$ be the first uncountable ordinal. A hierarchy of functions $\phi^{B_\alpha}(\beta) = 1 + \beta \phi^{B_{\alpha'}}(1) = (\phi^{B_\alpha'})'$ $\phi^{B_\lambda}$ enumerates $\bigcap_{\xi < \tau} \lambda_{\xi}$ of ordinals $\lambda_{\xi} \in B_\lambda$ of length $\tau_{\lambda} \leq B_\lambda$ and $\lim_{\xi < \tau} \lambda_{\xi} = \lambda_{\lambda}$.

$\phi^{B_\lambda}$ enumerates $\{ \beta < \Omega : \phi^{B_\lambda}(\beta)(0) = \beta \}$ of limit, $\tau_{\lambda} = \Omega$.
Bachmann’s novel idea: Use uncountable ordinals to keep track of the functions defined by diagonalization.

Define a set of ordinals $\mathcal{B}$ closed under successor such that with each limit $\lambda \in \mathcal{B}$ is associated an increasing sequence $\langle \lambda[\xi] : \xi < \tau_\lambda \rangle$ of ordinals $\lambda[\xi] \in \mathcal{B}$ of length $\tau_\lambda \leq \mathcal{B}$ and $\lim_{\xi<\tau_\lambda} \lambda[\xi] = \lambda$. 
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Let $\Omega$ be the first uncountable ordinal. A hierarchy of functions $(\varphi_\alpha^\mathcal{B})_{\alpha \in \mathcal{B}}$ is then obtained as follows:

\[
\begin{align*}
\varphi_0^\mathcal{B}(\beta) &= 1 + \beta \\
\varphi_{\alpha+1}^\mathcal{B} &= (\varphi_\alpha^\mathcal{B})' \\
\varphi_\lambda^\mathcal{B} &\text{ enumerates } \bigcap_{\xi < \tau_\lambda} (\text{Range of } \varphi_\lambda[\xi]) \lambda \text{ limit, } \tau_\lambda < \Omega \\
\varphi_\lambda^\mathcal{B} &\text{ enumerates } \{ \beta < \Omega : \varphi_\lambda[\beta](0) = \beta \} \lambda \text{ limit, } \tau_\lambda = \Omega.
\end{align*}
\]
After Bachmann, the story of ordinal representation systems becomes very complicated.

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- **Feferman**’s new proposal: Bachmann-type hierarchy without fundamental sequences.

- **Bridge** and **Buchholz** showed computability of systems obtained by Feferman’s approach.
“Natural” well-orderings

Set-theoretical (Cantor, Veblen, Gentzen, Bachmann, Schütte, Feferman, Pfeiffer, Isles, Bridge, Buchholz, Pohlers, Jäger, Rathjen)

• Define hierarchies of functions on the ordinals.
• Build up terms from function symbols for those functions.
• The ordering on the values of terms induces an ordering on the terms.

Reductions in proof figures (Takeuti, Yasugi, Kino, Arai)

• Ordinal diagrams; formal terms endowed with an inductively defined ordering on them.
Patterns of elementary substructurehood \((Carlson)\)

- Finite structures with \(\Sigma_n\)-elementary substructure relations.

Category-theoretical \((Aczel, Girard, Jervell, Vauzeilles)\)

- Functors on the category of ordinals (with strictly increasing functions) respecting direct limits and pull-backs.

Representation systems from below \((Setzer)\)
Second order arithmetic; $\mathbb{Z}_2$ aka Analysis

$\mathbb{Z}_2$ is a two sorted formal system. Extends PA.

- Variables $n, m, \ldots$ range over natural numbers.
- Variables $X, Y, Z, \ldots$ range over sets of natural numbers.
- Relation symbols $=, <, \in$. Function symbols $+, \times, \ldots$

Comprehension Principle/Axiom:
For any property $P$ definable in the language of $\mathbb{Z}_2$,
$\{n \in \mathbb{N} | P(n)\}$ is a set; or more formally
$(CA) \exists X \forall n [n \in X \leftrightarrow A(x)]$ for any formula $A(x)$ of $\mathbb{Z}_2$. 

FROM ARITHMETIC TO SET THEORY
**Second order arithmetic; Z₂ aka Analysis**

Z₂ is a two sorted formal system. Extends PA.

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  for any formula $A(x)$ of Z₂.
Stratification of Comprehension

- A $\Pi^1_k$-formula (or $\Sigma^1_k$-formula) is a formula of $\mathbb{Z}_2$ of the form

$$\forall X_1 \ldots QX_k A(X_1, \ldots, X_k) \quad (\exists X_1 \ldots QX_k A(X_1, \ldots, X_k))$$

with $\forall X_1 \ldots QX_k (\exists X_1 \ldots QX_k)$ a string of $k$ alternating set quantifiers, beginning with a universal quantifier (existential quantifier), followed by a formula $A(X_1, \ldots, X_k)$ without set quantifiers.
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- $\Pi^1_k$-comprehension ($\Sigma^1_k$-comprehension) is the scheme
  \[ \exists X \forall n [n \in X \leftrightarrow A(x)] \]
  with $A(x) \ \Pi^1_k \quad (\Sigma^1_k)$. 
Subsystems of \( \mathbb{Z}_2 \)

- Basic arithmetical axioms in all subtheories of \( \mathbb{Z}_2 \) are: defining axioms for 0, 1, +, \( \times \), \( E \), < (as for PA) and the induction axiom

\[
\forall X \left[ 0 \in X \land \forall n (n \in X \rightarrow n + 1 \in X) \rightarrow \forall n (n \in X) \right].
\]
Subsystems of $\mathbb{Z}_2$

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- $(\text{Ax})$ stands for the theory $(\text{Ax})_0$ augmented by the scheme of induction for all $\mathcal{L}_2$-formulae.
Subsystems of $\mathbb{Z}_2$

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- For each axiom scheme $\text{Ax}$, $\text{Ax}_0$ denotes the theory consisting of the basic arithmetical axioms plus the scheme $\text{Ax}$.
- $\text{Ax}_0$ stands for the theory $\text{Ax}_0$ augmented by the scheme of induction for all $L_2$-formulae.
- Let $\mathcal{F}$ be a collection of formulae of $\mathbb{Z}_2$.
  Another important axiom scheme for formulae $F$ in $\mathcal{C}$ is
  \[
  \mathcal{C} - \text{AC} \quad \forall n \exists Y F(n, Y) \rightarrow \exists Y \forall n F(x, Y_n),
  \]
  where $Y_n := \{ m : 2^n 3^m \in Y \}$. 

FROM ARITHMETIC TO SET THEORY
How much of $\mathbb{Z}_2$ is needed?

- **Hermann Weyl** 1918 “Das Kontinuum"  
  Predicative Analysis.
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How much of $\mathbb{Z}_2$ is needed?

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- Minimal foundational frameworks for Ordinary Mathematics:
  Feferman, Lorenzen, Takeuti ....
- **Reverse Mathematics**, early 1970s-now
  H. Friedman, S. Simpson, ....

Given a specific theorem $\tau$ of ordinary mathematics, which set existence axioms are needed in order to prove $\tau$?
For many mathematical theorems $\tau$, there is a weakest natural subsystem $S(\tau)$ of $\mathbb{Z}_2$ such that $S(\tau)$ proves $\tau$. Moreover, it has turned out that $S(\tau)$ often belongs to a small list of specific subsystems of $\mathbb{Z}_2$. Reverse Mathematics has singled out five subsystems of $\mathbb{Z}_2$:

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Mathematical Equivalences: Examples

- \( \text{RCA}_0 \)  “Every countable field has an algebraic closure”; “Every countable ordered field has a real closure"
Mathematical Equivalences: Examples

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  \quad \text{"Every countable ordered field has a real closure"}

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- **(Π\(_1\)₁ − CA)\(_0\)**  
  “Every uncountable closed set of real numbers is the union of a perfect set and a countable set";  
  “Every countable abelian group is a direct sum of a divisible group and a reduced group"
\[ |\mathbf{ATR}_0| = \Gamma_0 \]
\[ |\mathbf{ACA}_0| = \varepsilon_0 \]
\[ |\mathbf{RCA}_0| = \omega^\omega = |\mathbf{WKL}_0| \]
\[ |(\Sigma^1_2-\text{AC}) + \text{BI}| = \psi_{\Omega_1} \omega \]

\[ |(\Delta^1_2-\text{CA})| = \psi_{\Omega_1} \Omega_{\varepsilon_0} \]

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FROM ARITHMETIC TO SET THEORY
$$\left| \left( \Pi^1_2 - \text{CA} \right)_0 \right| = \psi_\Omega_1 R_\omega$$
A Brief History of Ordinal Analysis

- Gentzen 1936
  theory $\text{PA}$
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- Feferman, Schütte 1963
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• Gentzen 1936
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• Takeuti 1967
  $(\Pi^1_1\text{-CA})_0$, $(\Pi^1_1\text{-CA}) + \text{BI}$
  ordinals $\psi_{\Omega_1 \Omega_\omega}$, $\psi_{\Omega_1 \varepsilon_{\Omega_\omega} + 1}$
  cardinal analogue: $\omega$-many regular cardinals
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- Takeuti, Yasugi 1983
  $(\Delta^1_2\text{-CA})$
  ordinal $\psi_{\Omega_1\Omega_{\varepsilon_0}}$
  cardinal analogue: $\varepsilon_0$-many regular cardinals
Buchholz, Pohlers, Sieg 1977
Theories of Iterated Inductive Definitions
ordinals $\psi_{\Omega_1 \Omega_\nu}$
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A Brief History of Ordinal Analysis cont’d

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  $\Omega_{\nu+1}$-rules
A Brief History of Ordinal Analysis cont’d

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- Jäger 1979
  Constructible Hierarchy in Proof Theory
A Brief History of Ordinal Analysis cont’d

- Jäger, Pohlers 1982
  \((\Sigma^1_2\text{-}\text{AC}) + \text{BI, KPi}\)
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  cardinal analogue: \(I\) inaccessible cardinal

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  \(KPM\)
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From arithmetic to set theory
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  First-order reflection
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- **R 1995**
  \( \Pi^1_2 \)-Comprehension
  cardinal analogue: \( \omega \)-many reducible cardinals
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  ordinal $\psi_{\Omega_1}K$
  cardinal analogue: $K$ weakly compact cardinal

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  First-order reflection
  cardinal analogue: totally indescribable cardinal

- R 1995
  $\Pi_2^1$-Comprehension
  cardinal analogue: $\omega$-many reducible cardinals

- Arai
  Ordinal Analysis of Theories up to $\Pi_2^1$-Comprehension
  using Reductions on Finite Proof Figures and Ordinal Diagrams.