Set Theory: an Introduction to the World of Large Cardinals

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Joan Bagaria

ICREA Research Professor. Department of Logic and History and Philosophy of Science, University of Barcelona.

E-mail address: bagaria@.ub.edu

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Introduction

Large cardinals are infinite cardinal numbers κ that enjoy special combinatorial properties implying that they are very large and that V_κ is a model of the ZFC axioms of set theory, hence by Gödel's second incompleteness theorem their existence cannot be proved in ZFC. Most large cardinals can be characterized as cardinals that reflect a substantial amount of the structure of the universe V of all sets. For example, an inaccessible cardinal κ, the smallest kind of all large cardinals, can be characterized as being regular and such that V_κ reflects all existential statements, in the sense that if an existential statement involving sets in V_κ is true in V, then it is witnessed by a set in V_κ.

The combinatorial and reflective properties of large cardinals can be used for a variety of purposes. For example, for constructing mathematical objects (topological spaces, algebraic structures, etc.) with special properties, whose existence may not be provable in ZFC. Another use of large cardinals is to show that a given mathematical statement cannot be proved in ZFC by showing that the statement implies the existence of, or just the consistency of the existence of, some large cardinal. In fact, large cardinals provide a measure of the strength of formal systems beyond ZFC.

This course has few prerequisites. Some familiarity with first-order logic, the ZFC axioms, definitions by transfinite recursion, and a basic knowledge of ordinals and cardinals, which we shall review anyway in the Preliminaries section, should suffice. Everything else will be self-contained.

We will use standard set-theoretic notation. The basic bibliographical references are [**1**], Chapters 1-3, 5-10 of Part I; and [**2**], Chapter 1.

Preliminaries

1. The language of set theory

The formal *language of set theory* is the first-order language, with equality, whose only non-logical symbol is the binary relation symbol $\in$. The formulas of the language are defined recursively, as follows:

(1) *Atomic formulas* are of the form $x = y$ or $x \in y$.
(2) If φ and ψ are formulas, then so are $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, and $(\varphi \leftrightarrow \psi)$.
(3) If φ is a formula, then so are $\forall x\varphi$ and $\exists x\varphi$.

Parentheses may be added after a quantifier to facilitate the reading, and may be omitted if the formula can be read without ambiguity.

A variable is said to occur *free* in a formula if it does not fall within the range of any quantifier. Thus x occurs free in the formula $x \in y$, and so does y. The first occurrence of x in the formula $\forall y(x \in y) \wedge \exists x(\neg x \in z)$ is free, while the second is not, as it is *bound* by the existential quantifier.

A formula with no variables occurring free in it is called a *sentence*.

2. The ZFC axioms

We will work in the ZFC (Zermelo-Fraenkel with Choice) axiom system, which is the standard theory of sets. The axioms of Zermelo-Fraenkel are listed below. We state them both informally and formalized in the language of set theory. As is customary, we write $\forall x \in a\,(\ldots)$ for $\forall x(x \in a \to \ldots)$, and $\exists x \in a\,(\ldots)$ for $\exists x(x \in a \wedge \ldots)$. The actual formal axioms are the universal closure of the displayed formulas.

Extensionality: If two sets a and b have the same elements, then they are equal.

$$\forall x(x \in a \leftrightarrow x \in b) \to a = b$$

Pair: Given any sets a and b, there exists a set containing a and b as elements.

$$\exists x(a \in x \wedge b \in x)$$

Union: For every set a, there is a set containing all elements of the elements of a.

$$\exists x \forall y \in a \forall z \in y(z \in x)$$

Power set: For every set a there is a set that contains all subsets of a.

$$\exists x \forall y (\forall z \in y (z \in a) \to y \in x)$$

Infinity: There exists an infinite set.

$$\exists x (\exists y (y \in x) \wedge \forall y \in x \exists z \in x (y \in z))$$

Foundation: Every non-empty set a contains an $\in$-minimal element.

$$\exists y (y \in a) \to \exists y \in a \forall z \in a (z \notin y)$$

Separation: For every set a and every *property*, there is a set containing exactly the elements of a that have this *property*.

$$\exists x \forall y (y \in x \leftrightarrow y \in a \wedge \varphi(y))$$

for every formula $\varphi(y)$ of the language of set theory in which x does not occur free and which may have other free variables. So this is an infinite list of axioms, one for each such formula $\varphi(y)$.

Replacement: For every *definable (multivalued) function* on a set a, there is a set containing all the values of the *function*.

$$\forall x \in a \exists y \varphi(x, y) \to \exists z \forall x \in a \exists y \in z \, \varphi(x, y)$$

for every formula $\varphi(x, y)$ of the language of set theory in which z does not occur free and which may have other free variables. This is also an infinite list of axioms, one for each such formula $\varphi(x, y)$.

Observe that Separation follows easily from Replacement.

The Axiom of Choice (AC) is the following:

Choice: For every set a of pairwise disjoint non-empty sets, there exists a set that contains exactly one element from each set in a.

AC is equivalent, modulo the Zermelo-Fraenkel axioms, to Zermelo's *Well-Ordering Principle*: Every set can be well-ordered. That is, for every set a there exists an ordering relation on a that is a well-order. (Recall that a well-order of a is a linear ordering of a in which every non-empty subset of a has a least element.)

Another useful equivalent form of AC is *Zorn's Lemma* (Hausdorff 1914): if $\langle \mathbb{P}, \leq \rangle$ is a partially-ordered set in which every linearly-ordered subset has an upper bound in $\mathbb{P}$, then there is a maximal element, i.e., some $p \in \mathbb{P}$ such that for no $q \in \mathbb{P}$ we have $p < q$.

3. Sets versus proper classes

Some collections are not sets. For example, the collection of all sets, V, is not a set. Otherwise, by the Separation axiom, there exists a set $A =: \{x \in V : x \notin x\}$. But then $A \in A$ if and only if $A \notin A$. This is known as Russell's Paradox.

Collections that are not sets are called *proper classes*. In ZFC, proper classes are given by a formula, as in the previous example A was given by the formula $x \notin x$.

4. Ordinals

A set A is *transitive* if it contains all elements of its elements.

An *ordinal number*, or simply an *ordinal*, is a transitive set well-ordered by $\in$. The empty set $\emptyset$ is an ordinal.

If α and β are ordinal numbers, then $\alpha \in \beta$ if and only if $\alpha \subset \beta$. Thus, $\alpha \in \beta$ if and only if α is a proper $\in$-initial segment of β. It follows that every ordinal α is precisely the set of all its $\in$-predecessors, which are themselves ordinals. We usually write $\alpha < \beta$ for $\alpha \subset \beta$, and $\alpha \leq \beta$ for $\alpha \subseteq \beta$. Thus, for all ordinal numbers α and β, either $\alpha < \beta$, or $\beta < \alpha$, or $\alpha = \beta$.

If α is an ordinal, then so is $\alpha \cup \{\alpha\}$. And if X is a set of ordinals, then $\bigcup X$ is also an ordinal. The ordinals form a proper class, denoted by Ω or OR, which is well-ordered by $\leq$.

The *(immediate) successor* of an ordinal α is the ordinal $\alpha \cup \{\alpha\}$, usually denoted by $\alpha + 1$. A *limit ordinal* is an ordinal that is neither empty nor a successor.

The *natural numbers* are identified with the finite ordinals. Thus, $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, and so on. The set $\mathbb{N}$ of natural numbers is thus identified with the first infinite ordinal number, which is also the first limit ordinal, and is denoted by ω.

An ordinal is *countable* if it is either finite or bijectable with ω. The set of all countable ordinals is not countable and is, therefore, the first uncountable ordinal, denoted by ω_1. The set of all ordinals bijectable with some $\alpha \leq \omega_1$ is an ordinal not bijectable with any $\alpha \leq \omega_1$ and is denoted by ω_2. And so on.

A limit ordinal α is called *regular* if there is no function $f : \beta \to \alpha$ with $\beta < \alpha$ and $range(f)$ unbounded in α. Otherwise, α is called *singular*. The *cofinality* of α (denoted by $cof(\alpha)$) is the least $\beta \leq \alpha$ for which there exists $f : \beta \to \alpha$ with range *cofinal*, i.e., unbounded, in α. Thus, α is regular if and only if $cof(\alpha) = \alpha$. Notice that $cof(\alpha)$ is a regular ordinal, for every limit ordinal α.

All the ordinals $\omega, \omega_1, \omega_2, \ldots$ are regular. The limit of all these, that is, $\bigcup_n \omega_n$, is a singular ordinal, denoted by ω_ω.

By the Well-Ordering Principle, every set can be well-ordered. And every well-ordered set X is order-isomorphic to a unique ordinal, denoted by $otp(X)$, the *order-type* of X.

5. The universe of all sets

In ZFC, one can prove that the universe of all sets V forms a *cumulative hierarchy*. That is, every set belongs to some V_α, for some ordinal α, where the V_α are defined as follows:

$V_0 = \emptyset$

$V_{\alpha+1} = \mathcal{P}(V_\alpha)$, the power set of V_α.

$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$, if λ is a limit ordinal.

Then, $V = \bigcup_{\alpha\in\Omega} V_\alpha$ is the universe of all sets.

Notice that $\alpha \leq \beta$ implies $V_\alpha \subseteq V_\beta$.

One can easily see, by transfinite induction on the ordinals α, that all the V_α are transitive sets.

6. Cardinals

A *cardinal number* (or simply, a *cardinal*) is an ordinal that is not bijectable with any smaller ordinal. Thus, all natural numbers are cardinals, and so are ω, ω_1, $\omega_2, \ldots, \omega_\omega, \ldots$

Every infinite cardinal is a limit ordinal.

We normally use Greek letters $\kappa, \lambda, \mu, \nu, \ldots$ to denote infinite cardinals.

Given an infinite cardinal κ, the set of all ordinals that are bijectable with some $\lambda \leq \kappa$ is a cardinal; it is the least cardinal greater than κ, and is usually denoted by κ^+. Moreover, if X is a set of cardinals, then $\bigcup X$ is also a cardinal. Hence, the cardinals form a proper class contained in Ω. The transfinite sequence of all infinite cardinals is denoted, following Cantor, by the Hebrew letter $\aleph$ (aleph) sub-indexed by ordinals. Thus,

$$\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_\omega, \aleph_{\omega+1}, \ldots, \aleph_\alpha, \ldots$$

Notice that $\aleph_n = \omega_n$, for all $n < \omega$.

The Well-Ordering Principle implies that every set has a *cardinality*, i.e., is bijectable with a (unique) cardinal $\aleph_\alpha$. The cardinal $\aleph_\alpha$ is called the cardinality of X and is denoted by $|X|$.

Exercise 6.1.

(1) *If α is a limit ordinal, then $cof(\aleph_\alpha) = cof(\alpha)$.*
(2) *If κ is an infinite cardinal, then κ^+ is regular.*

6.1. Some cardinal arithmetic.

Let κ, λ be cardinals.

The sum $\kappa + \lambda$ is defined as $|A \cup B|$, for some sets A and B with $|A| = \kappa$, $|B| = \lambda$, and $A \cap B = \emptyset$. Equivalently, as $|\kappa \times \{0\} \cup \lambda \times \{1\}|$.

The product $\kappa \cdot \lambda$ is defined as $|\kappa \times \lambda|$.

The exponentiation is defined as $\kappa^\lambda = |\prod_{\alpha<\lambda} \kappa|$, i.e., the cardinality of the product of λ-many copies of κ. Equivalently, the cardinality of the set of all functions from λ into κ.

Since for every infinite cardinal κ the canonical pairing function on the ordinals (see [**1**]) is a bijection between $\kappa \times \kappa$ and κ, it follows that $\kappa \cdot \kappa = \kappa$, and therefore for all infinite cardinals κ and λ,

$$\kappa + \lambda = \kappa \cdot \lambda = max\{\kappa, \lambda\}.$$

So the sum and product of infinite cardinals is trivial. However, the exponentiation is, in contrast, highly non-trivial. Indeed, even the value of $2^{\aleph_0}$ cannot be decided in ZFC.

If $2 \leq \kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$, because $2^\lambda \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^{\kappa \cdot \lambda} = 2^\lambda$.

Cantor's Theorem states that $|A| > |\mathcal{P}(A)|$, for every set A. Hence, $2^\kappa > \kappa$, for every cardinal κ.

Another result one can prove in ZFC about infinite cardinal exponentiation is that $\kappa^{cof(\kappa)} > \kappa$, for every infinite cardinal κ.

But, unfortunately, this is about all one can prove in ZFC in such a generality about cardinal exponentiation, assuming of course that ZFC is consistent.

7. Models, consistency, and independence

Since ZFC is a recursive axiom system in which arithmetic is formalizable, it is subject to Gödel's Second Incompleteness Theorem. Namely, if ZFC is *consistent*, i.e., no contradiction can be logically derived from it, then ZFC cannot prove its own consistency. However, we do believe ZFC is consistent, since all ZFC axioms are true in V.

A *structure* for the language of set theory is a pair $\langle M, E\rangle$, where M is a set or a proper class and E is a binary relation on M. We say that $\langle M, E\rangle$ is a *model* of ZFC if all ZFC axioms are *true in* $\langle M, E\rangle$ whenever we interpret the variables as ranging over elements of M and we interpret $\in$ as E. We sometimes consider also models of fragments of ZFC.

Exercise 7.1. *Show that the pair $\langle \omega, E\rangle$, where E is the relation given by: mEn iff* the m-th digit (counting from right to left) in the binary expansion of n is 1*, is a model of ZFC minus Infinity. In fact, $\langle \omega, E\rangle$ and $\langle V_\omega, \in\rangle$ are isomorphic.*

By Gödel's Completeness Theorem for first-order logic, ZFC has a model if and only if it is consistent. Hence, by Gödel's Second Incompleteness Theorem, if ZFC is consistent, then one cannot prove in ZFC that there exists a model of ZFC.

A model $\langle M, E\rangle$ is called *standard* if E is $\in$, that is, the membership relation between sets. Namely, $E = \in \cap (M \times M)$. If $\langle M, E\rangle$ is standard, then we usually write $\in$ instead of E, or we just write M instead of $\langle M, E\rangle$. Thus, V is a standard proper class model of ZFC.

The main reason for building models of ZFC of various sorts is to prove consistency and independence results in mathematics. For suppose φ is a mathematical statement. Since virtually every mathematical statement can, in principle, be translated into the language of set theory, we may assume φ is in fact a sentence in that language. Now suppose we can build a model of ZFC (or of an arbitrarily large finite fragment of ZFC) where φ holds. Then the negation of φ is not provable in ZFC. The reason is that in any purported proof of the negation of φ only a finite number of axioms of ZFC would be used, but then in every model of those axioms φ would be false. Similarly, if we can build a model of ZFC (or of an arbitrarily large finite fragment of ZFC) in which the negation of φ holds, then φ is not provable in ZFC.

Thus, considering that *being formally provable in ZFC* is a widely accepted proper mathematical rendition of *being provable using the methods usually available in mathematics*, it is clear that building models of (fragments of) ZFC where a given mathematical statement holds is of great interest, for it provides a mathematical proof that the statement cannot be refuted using the usual mathematical tools.

A sentence φ is said to be *independent of ZFC* if neither φ not its negation are provable in ZFC. Equivalently, if there exist two models of ZFC, one that satisfies φ and and one that satisfies its negation.

The most famous example of independence of ZFC is Cantor's Continuum Hypothesis (CH). Georg Cantor formulated in 1874 the hypothesis that every infinite set of real numbers is either countable (i.e., it can be put into a one-to-one correspondence with the natural numbers) or it has the same cardinality as $\mathbb{R}$ (i.e., it can be put into one-to-one correspondence with the real numbers). This is equivalent to saying that the cardinality of $\mathbb{R}$ is $\aleph_1$, and also equivalent to $2^{\aleph_0} = \aleph_1$.

The CH was Hilbert's number one problem in his famous list of unsolved mathematical problems he presented at the second International Congress of Mathematicians, held in Paris in 1900. In spite of many attemps by Cantor himself and others to prove CH, it was not until 60 years later, in 1938, that Gödel was able to construct his model L, the universe of constructible sets, and proved that CH holds in it, thereby showing that CH is consistent with ZFC. Further, in 1963, Paul Cohen invented a new revolutionary and extremely powerful method for expanding a given model of ZFC, called *forcing*, and used it to obtain models of ZFC in which CH fails, thereby showing that the negation of CH is also consistent with ZFC.

The *Generalized Continuum Hypothesis (GCH)* states that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, for all $\alpha \in \Omega$. The GCH is also independent of ZFC.

7.1. Consistency strength. If φ is a sentence of the language of set theory, let $CON(\varphi)$ be the Π_1 arithmetical formula that asserts that there is no proof of $\neg\varphi$ from ZFC. Notice that ZFC, being a recursive set, has a Δ_1 definition (i.e., both a Σ_1 definition and a Π_1 definition) in $\langle \omega, +, \cdot, 0, 1 \rangle$.

By the completeness theorem for first-order logic,

$$ZFC \vdash (\langle\omega, +, \cdot, 0, 1\rangle \models CON(\varphi) \leftrightarrow \exists M(M \models ZFC + \varphi)).$$

Let φ and ψ be sentences of the language of set theory. We say that φ *has higher consistency strength than* ψ *(modulo ZFC)*, or that φ is *consistency-wise stronger than* ψ *(modulo ZFC)* if

$$ZFC \vdash (CON(\varphi) \rightarrow CON(\psi))$$

but

$$ZFC \nvdash (CON(\psi) \rightarrow CON(\varphi)).$$

Equivalently,

$$ZFC \vdash (\exists M(M \models ZFC + \varphi) \rightarrow \exists M(M \models ZFC + \psi))$$

but

$$ZFC \nvdash (\exists M(M \models ZFC + \psi) \rightarrow \exists M(M \models ZFC + \varphi)).$$

We say that φ and ψ *have the same consistency strength (modulo ZFC)*, or that φ and ψ are *equiconsistent (modulo ZFC)* if

$$ZFC \vdash (CON(\varphi) \leftrightarrow CON(\psi)).$$

Equivalently,

$$ZFC \vdash (\exists M(M \models ZFC + \varphi) \leftrightarrow \exists M(M \models ZFC + \psi)).$$

Of course, equivalence (modulo ZFC) implies equiconsistency (modulo ZFC).

8. The Mostowski collapse

A binary relation E on a set or a proper class X is *well-founded* if the following two conditions hold:

(1) There is no infinite descending E-chain

$$\dots a_{n+1} E a_n \dots a_2 E a_1 E a_0.$$

Equivalently, every non-empty subset of X has an E-minimal element.

(2) For every $x \in X$, the collection of all $y \in X$ such that yEx is a set. (This, of course, holds automatically if X itself is a set.)

If E is a well-founded relation on a set (or a proper class) X, then the *rank* function

$$\rho(x) = sup\{\rho(y) + 1 : yEx\} \tag{1}$$

maps X onto an ordinal, or onto Ω if X is a proper class, and is order-preserving, i.e., xEy implies $\rho(x) < \rho(y)$. To see this, let $X_0 = \emptyset$ and let $X_{\alpha+1} = X_\alpha \cup \{x \in X : \forall y(yEx \rightarrow y \in X_\alpha\}$. For λ a limit ordinal, let $X_\lambda = \bigcup_{\alpha<\lambda} X_\alpha$. Notice that the X_α form an increasing chain, i.e., $\alpha < \beta$ implies $X_\alpha \subseteq X_\beta$. Now one can easily check that $\rho(x)$ is the least α such that $x \in X_{\alpha+1}$. Hence, by Replacement there is γ such that $X_\gamma = X_{\gamma+1}$, in which case $X_\gamma = X$ (or $X_\Omega = X$ if X is a proper class).

The function ρ is the unique function satisfying equation 1 above, that is, if ρ' is another such function, then $\rho = \rho'$. Otherwise, let α be the least ordinal such that the set $\{x \in X_\alpha : \rho(x) \neq \rho'(x)\}$ is non-empty, and let x be an E-minimal element in this set. By minimality of α and x, we have $\rho(y) = \rho'(y)$, for all yEx. But then we must have $\rho(x) = \rho'(x)$, which is impossible.

For each $x \in X$, $\rho(x)$ is called the *rank* (*E-rank*) of x.

Suppose E is a well-founded relation on X. We call a subset x of X *E-transitive* if for every $y \in x$, if zEy, then $z \in x$.

THEOREM 8.1 (Transfinite recursion on well-founded relations). *Suppose E is a well-founded relation on a class X. If G is a class function defined on V, then there is a unique class function F on X such that*

$$F(x) = G(x, F \restriction \{z : zEx\}).$$

PROOF. (Sketch) Define F as follows:

$F(x) = y$ if and only if there is a function f with domain an E-transitive set containing x such that for every z in the domain of f, $f(z) = G(z, f \restriction \{t : tEz\})$ and $f(x) = y$.

By induction on $\alpha \geq 1$ one can check that F is defined for all $x \in X_\alpha$.

Uniqueness follows by considering another such F', looking at the least α such that the set $\{x \in X_\alpha : F(x) \neq F'(x)\}$ is non-empty, and then taking an E-minimal element x in this set. It follows that $F(x) = F'(x)$, yielding a contradiction. □

A model $\langle M, E\rangle$ is called *well-founded* if E is well-founded on M.

THEOREM 8.2 (Mostowski Collapse). *If $\langle M, E\rangle$ is a well-founded model of the axiom of Extensionality, then there is a unique transitive model $\langle N, \in\rangle$ (called the* transitive, or Mostowski, collapse *of $\langle M, E\rangle$) and a unique isomorphism $\pi : \langle M, E\rangle \to \langle N, \in\rangle$.*

PROOF. Let $\pi(x) = \{\pi(z) : zEx\}$. Clearly, aEb implies $\pi(a) \in \pi(b)$. So we only need to check that $\pi(a)$ exists for every $a \in M$, and that π is one-to-one. Then we can take N to be the range of π.

Existence is guaranteed by Theorem 8.1 above. Indeed, consider the function G such that for each function f with domain an E-transitive set containing x, assigns to the pair $(x, f \restriction \{z : zEx\})$ the set $\{f(z) : zEx\}$. Then $\pi(x) = G(x, \pi \restriction \{z : zEx\})$.

We can see that π is one-to-one by induction on the E-rank ρ of the elements of M. Since M is a model of Extensionality, there is only one element a of M having E-rank 1, and then $\pi(a) = \emptyset$. Now suppose $a, b \in M$ and $a \neq b$. Since M satisfies Extensionality, we can find, say, some cEa such that $\neg cEb$. Hence, $\pi(c) \in \pi(a)$. We claim that $\pi(c) \notin \pi(b)$, and therefore $\pi(a) \neq \pi(b)$. Otherwise, there is dEb with $\pi(c) = \pi(d)$. But since c, d are of lower rank than b, and they are different, by inductive hypothesis we have $\pi(c) \neq \pi(d)$. □

The following is a typical application of the Mostowski collapse. Suppose $\langle X, \in \rangle$ is a model of a fragment T of ZFC. By the Löwenheim-Skolem Theorem, let $\langle M, \in \rangle$ be a countable elementary substructure of $\langle X, \in \rangle$. Apply the Mostowski collapse to $\langle M, \in \rangle$ to obtain a countable transitive model $\langle N, \in \rangle$ isomorphic to $\langle M, \in \rangle$. Then we have that $\langle N, \in \rangle$ is a model of T.

It follows that for every finite fragment T of ZFC there is a countable transitive model of the form $\langle N, \in \rangle$ that satisfies T.

9. Filters

Recall the notions of *filter* on a set.

DEFINITION 9.1. *A* filter *on a non-empty set A is a set $\mathcal{F}$ of subsets of A such that:*

(1) *$A \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.*
(2) *If $X, Y \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.*
(3) *If $X \in \mathcal{F}$ and $X \subseteq Y \subseteq A$, then $Y \in \mathcal{F}$.*

If $\mathcal{F}$ is a filter on A, then $\{A - X : X \in \mathcal{F}\}$ is an *ideal* on A, called the *dual ideal of $\mathcal{F}$*.

9.1. The filter of closed unbounded sets. A subset C of an infinite ordinal α is *unbounded* if for every $\beta < \alpha$ there is $\gamma \in C$ greater than β. And C is *closed* if the supremum of every increasing sequence of elements of C belongs to C, provided this supremum is $< \alpha$. Thus, C is closed if and only if for every limit ordinal $\beta < \alpha$, if $C \cap \beta$ is unbounded in β, then $\beta \in C$. We say that C is a *club* subset of α if it is closed and unbounded.

If κ is an uncountable cardinal, then the set of limit ordinals smaller than κ is club And if λ is a limit cardinal, then the set of cardinals smaller than λ is club.

PROPOSITION 9.2. *If α is an infinite ordinal of uncountable cofinality, then the set $Club(\alpha) := \{X \subseteq \alpha : C \subseteq X,$ for some club $C\}$ is a filter, called the club filter on α.*

PROOF. We only need to check that the intersection of any two sets in $Club(\alpha)$ is in $Club(\alpha)$. This follows immediately from the fact that the intersection of any two club sets is club. For suppose C and D are club. Given $\beta < \alpha$, pick alternatively $\gamma_{2n} \in C$ and $\gamma_{2n+1} \in D$ so that

$$\beta < \gamma_0 < \gamma_1 < \ldots < \gamma_{2n} < \gamma_{2n+1} < \ldots$$

Then, $sup\{\gamma_{2n} : n < \omega\} = sup\{\gamma_{2n+1} : n < \omega\} \in C \cap D$, because C and D are closed and α has uncountable cofinality. This shows $C \cap D$ is unbounded. That $C \cap D$ is also closed follows immediately from the fact that both C and D are closed. □

THEOREM 9.3. *If κ is a regular uncountable cardinal, then $Club(\kappa)$ is κ-complete, i.e., the intersection of less than κ-many club sets is club.*

Proof. Let $\langle C_\alpha : \alpha < \lambda\rangle$, with $\lambda < \kappa$, be a sequence of club sets subsets of κ. We will prove that $\bigcap_{\alpha<\lambda} C_\alpha$ is club by induction on λ.

We already saw that the intersection of two club sets is club. So we only need to consider the case λ is a limit and assume that the intersection of every sequence of length less than λ of club sets is club.

By taking $\bigcap_{\beta\leq\alpha} C_\beta$ instead of C_α, we may assume that the sequence of C_α's is decreasing, i.e., $C_\beta \supseteq C_\alpha$ whenever $\beta \leq \alpha$.

Let $C = \bigcap_{\alpha<\lambda} C_\alpha$. Clearly C is closed, since so are all the C_α's. Thus we only need to check that C is unbounded. So fix $\beta < \kappa$. Now define a sequence $\langle \beta_\alpha : \alpha < \lambda\rangle$ as follows: $\beta_0 = \beta$; $\beta_{\alpha+1}$ is the least ordinal in C_α greater than β_α (this is possible because C_α is unbounded); and if α is a limit, then take β_α to be the least ordinal in C_α greater than $sup\{\beta_\gamma : \gamma < \alpha\}$ (this is possible because κ is regular). Then $sup\{\beta_\alpha : \alpha < \lambda\} \in C$. □

Of course, it is not the case that the intersection of κ-many club sets is club. But the diagonal intersection is. Let κ be a regular uncountable cardinal. Given a sequence $\langle X_\alpha : \alpha < \kappa\rangle$ of subsets of κ, the *diagonal intersection* $\Delta_{\alpha<\kappa} X_\alpha$ is defined as the set $\{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} X_\beta\}$.

Proposition 9.4. *If κ is a regular uncountable cardinal and $\langle C_\alpha : \alpha < \kappa\rangle$ is a sequence of club subsets of κ, then $\Delta_{\alpha<\kappa} C_\alpha$ is club.*

Proof. Notice first that we may replace C_α by $D_\alpha := \bigcap_{\beta\leq\alpha} C_\beta$, because $\Delta_{\alpha<\kappa} C_\alpha = \Delta_{\alpha<\kappa} D_\alpha$. By Theorem 9.3 all the D_α are club. Note that the sequence of the D_α is decreasing, i.e., $D_\alpha \supseteq D_\beta$ for all $\alpha < \beta < \kappa$.

Now let $C = \Delta_{\alpha<\kappa} D_\alpha$ and let us show that C is club. Suppose first that $\alpha < \kappa$ is a limit point of C. If $\beta < \alpha$, then every $\gamma \in C$ such that $\beta \leq \gamma < \alpha$ belongs to D_β. Hence since D_β is closed, $\alpha \in D_\beta$. Therefore, $\alpha \in C$.

To see that C is unbounded, fix $\alpha < \kappa$. Construct a sequence $\{\beta_n : n < \omega\}$ as follows. Let $\beta_0 \in D_0$ be greater than α. Given β_n, pick $\beta_{n+1} > \beta_n$ in D_{β_n}. Then let β be the limit of the β_n. We claim that $\beta \in C$. For this it is enough to see that $\beta \in D_\gamma$ for all $\gamma < \beta$. If $\gamma < \beta$, let n be such that $\gamma < \beta_n$. But each β_m, for $m > n$, belongs to D_{β_n}, and so $\beta \in D_{\beta_n} \subseteq D_\gamma$. □

10. Stationary sets

The dual of the club filter on a cardinal κ of uncountable cofinality is the ideal NS_κ of non-stationary sets.

A subset S of κ is called *stationary* if it intersects all club subsets of κ. Thus, every club set is stationary. Moreover, if S is stationary and C is club, then $S \cap C$ is stationary.

By duality, it follows from Proposition 9.3 that if κ is regular and uncountable, then NS_κ is κ-complete, that is, the union of less than κ-many non-stationary sets is non-stationary.

There are many stationary sets that are not club.

PROPOSITION 10.1. *If* $\lambda < cof(\kappa)$ *is a regular cardinal, then the set* $E^\kappa_\lambda := \{\alpha < \kappa : cof(\alpha) = \lambda\}$ *is stationary.*

PROOF. Let C be a club subset of κ. Since $\lambda < cof(\kappa)$, the λ-th element α of C is less than κ, and since λ is regular α has cofinality λ. □

Thus, for example, the set $E^{\omega_2}_\omega$ is a stationary subset of ω_2 that is not closed. However, $E^{\omega_1}_\omega$ is closed, for it is the set of all countable limit ordinals. Notice that $E^{\omega_2}_\omega$ and $E^{\omega_2}_{\omega_1}$ are disjoint not-closed stationary subsets of ω_2.

A function f on a set of ordinals A is called *regressive* if $f(\alpha) < \alpha$ for every $\alpha \in A$, $\alpha > 0$.

The following theorem is known as the Pressing-Down Lemma, and also as Fodor's Lemma.

THEOREM 10.2. *Let* κ *be a regular uncountable cardinal, and let* $S \subseteq \kappa$ *be stationary. If* $f : S \to \kappa$ *is regressive, then there is a stationary* $S' \subseteq S$ *on which* f *is constant.*

PROOF. Suppose, towards a contradiction, that for every $\alpha < \kappa$, the set $\{\beta \in S : f(\beta) = \alpha\}$ is not stationary. So let $C_\alpha \subseteq \kappa$ be club and disjoint form the set. Thus, $f(\beta) \neq \alpha$ for every $\beta \in S \cap C_\alpha$. Now let $C = \Delta_{\alpha<\kappa} C_\alpha$. Then $S \cap C$ is stationary and if $\beta \in S \cap C$, then $f(\beta) \neq \alpha$ for all $\alpha < \beta$, contradicting the fact that f is regressive on S. □

THEOREM 10.3 (R. Solovay, 1971). *If* κ *is a regular uncountable cardinal, then* κ *can be partitioned into* κ*-many disjoint stationary sets.*

PROOF. If κ is a regular limit cardinal, then there are κ-many regular cardinals smaller than κ. By Proposition 10.1, the sets E^κ_λ, for regular $\lambda < \kappa$, are stationary and pairwise-disjoint.

So assume $\kappa = \lambda^+$, for some λ. For each $\alpha < \kappa$, let f_α be a one-to-one function from α into λ. Now for each $\beta < \kappa$ and each $\gamma < \lambda$, let

$$X^\gamma_\beta = \{\alpha > \beta : f_\alpha(\beta) = \gamma\}.$$

If $\beta \neq \beta'$, then $X^\gamma_\beta \cap X^\gamma_{\beta'} = \emptyset$, because the f_α's are one-to-one. Moreover, for each $\beta < \kappa$,

$$\bigcup_{\gamma<\lambda} X^\gamma_\beta = \{\alpha : \beta < \alpha < \kappa\}.$$

Since NS_κ is κ-complete, at least one of the X^γ_β's is stationary. So for each $\beta < \kappa$, let $g(\beta)$ be such that $X^{g(\beta)}_\beta$ is stationary. Since $g : \kappa \to \lambda$, and κ is regular, there is γ such that $X := \{\beta : g(\beta) = \gamma\}$ has cardinality κ. Thus, $\{X^\gamma_\beta : \beta \in X\}$ is a family of pairwise-disjoint stationary sets, as required. □

11. The Levy hierarchy of formulas

A formula in a first-order language that contains the language of set theory is Σ_0, or Π_0, if has only bounded quantifiers $\forall x \in y$ and $\exists x \in y$.

A formula is Σ_1 if it is of the form

$$\exists x_0, \ldots, x_k \varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$$

where $\varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$ is Π_0.

A formula is Π_1 if it is of the form

$$\forall x_0, \ldots, x_k \varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$$

where $\varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$ is Σ_0.

In general, a formula is Σ_n, $n > 1$ if it is of the form

$$\exists x_0, \ldots, x_k \varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$$

where $\varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$ is Π_{n-1}.

And a formula is Π_n, $n > 1$, if it is of the form

$$\forall x_0, \ldots, x_k \varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$$

where $\varphi(x_0, \ldots, x_k, y_0, \ldots, y_l)$ is Σ_{n-1}.

Σ_1 formulas are *upwards absolute* for transitive sets or classes. That is, if $M \subseteq N$ are transitive, $\varphi(x)$ is a Σ_1 formula, and $a \in M$ is such that $\varphi(a)$ is true in M, written $M \models \varphi(a)$, then $N \models \varphi(a)$. (Exercise.) Similarly, Π_1 formulas are *downwards absolute* for transitive sets or classes, that is, if $\varphi(x)$ is Π_1, $a \in M$, and $N \models \varphi(a)$, then $M \models \varphi(a)$.

If in a formula $\varphi(x_0, \ldots, x_k)$, where $x_0, \ldots, x_k$ occur free, we fix the values $a_0, \ldots, a_k$ of the variables $x_0 \ldots, x_k$, then we say that $\varphi(a_0, \ldots, a_k)$ is a formula with *parameters* $a_0, \ldots, a_k$.

11.1. Truth definition. We assume we have some fixed primitive recursive coding by natural numbers of the syntax of the language of set theory so that the arithmetical predicates x *codes a formula*, x *codes a sentence*, x *codes a* Σ_n*-formula*, etc. are Δ_1-definable in ZFC. This means, e.g., that there is a Σ_1 formula (and also a Π_1 formula) $\varphi(x)$ such that for every natural number n, if n codes a formula, then $ZFC \vdash \varphi(n)$, and if n does not code a formula, then $ZFC \vdash \neg\varphi(n)$. Therefore, assuming ZFC is consistent, n codes a formula iff $ZFC \vdash \varphi(n)$.

Notice, however, that we may assume that formulas themselves are sets, since they are finite sequences of simbols of the language of set theory, which we may assume they are, e.g., natural numbers, and therefore they are sets. Hence there is no real need for coding, for if the set of variables of the language of set theory is chosen to be, say, a recursive set of natural numbers, then the set of formulas is Δ_1 definable in ZFC. We shall write, e.g., $\varphi(\bar{x}) \in \Sigma_n$ to mean that $\varphi(\bar{x})$ is a Σ_n formula.

It follows from Tarski's theorem on the undefinability of truth that there is no formula of the first-order language of set theory that defines truth in ZFC, or, in fact, in any other theory that interprets primitive recursive arithmetic. i.e., there is no formula $\varphi(x)$ such that for every natural number n, n codes a theorem of ZFC iff $ZFC \vdash \varphi(n)$. Indeed, if $\varphi(x)$ were such a formula, let $\{\psi_k(x) : k < \omega\}$ be a primitive recursive enumeration of all formulas with one free variable. Let $\theta(x)$ be the formula $x \in \omega \wedge \neg\varphi(\ulcorner \psi_x(x) \urcorner)$, where for a formula ψ, $\ulcorner \psi \urcorner$ is Gödel's notation for the code of ψ. Let k be such that $\theta(x) = \psi_k$ and let σ be $\theta(k)$. Then, σ iff $\theta(k)$ iff $\neg\varphi(\ulcorner \psi_k(k) \urcorner)$ iff $\neg\varphi(\ulcorner \sigma \urcorner)$. Thus, we have that σ is a theorem of ZFC iff $ZFC \vdash \neg\varphi(\ulcorner \sigma \urcorner)$.

However, there is a Δ_1-definition of truth for Σ_0 sentences, namely, let $\models^0$ be the binary relation $\models^0 (\psi(\bar{x}), \bar{a})$ iff

(∗) $\psi(\bar{x})$ is a Σ_0 formula with k free variables, $\bar{a}$ is a k-tuple and *there exists* M transitive such that $\bar{a} \in M$ and $M \models \psi(\bar{a})$.

Since Σ_0 sentences are absolute for transitive sets, we also have $\models^0 (\psi(\bar{x}), \bar{a})$ iff

(∗∗) $\psi(\bar{x})$ is a Σ_0 formula with k free variables, $\bar{a}$ is a k-tuple and *for all* M transitive such that $\bar{a} \in M$, $M \models \psi(\bar{a})$.

Thus, since the satisfaction relation for sets $Sat(M, \psi(\bar{x}), \bar{a})$ (i.e., $Sat(M, \psi(\bar{x}), \bar{a})$ iff $\psi(\bar{x})$ is a formula with k free variables, $\bar{a}$ is a k-tuple of elements of M and $M \models \psi(\bar{a})$) is Δ_1, (∗) is Σ_1 and (∗∗) is Π_1. Hence, $\models^0$ is Δ_1.

For every Σ_0 formula $\psi(\bar{x})$, the following is provable in ZFC:

$$\forall \bar{a}(\psi(\bar{a}) \leftrightarrow \models^0 (\psi(\bar{x}), \bar{a})).$$

In particular, for every natural number n that codes a Σ_0 sentence ψ we have that n codes a theorem of ZFC iff $ZFC \vdash \models^0 (\psi, \emptyset)$.

Moreover, for every natural number $n \geq 1$, there is a Σ_n-definition of truth for Σ_n sentences, namely: $\models^n (\varphi(\bar{x}), \bar{a})$ iff

$\varphi(\bar{x})$ is a Σ_n formula with k free variables, $\varphi(\bar{x}) = \exists y_n \forall y_{n-1} .... \mathcal{Q}_1 y_1 \psi(\bar{x}, \bar{y})$, where $\psi(\bar{x}, \bar{y})$ is a restricted formula , $\bar{a}$ is a k-tuple and $\exists y_n \forall_{y-1} .... \mathcal{Q}_1 y_1 \models^0 (\psi(\bar{x}, \bar{y}), k+n, \bar{a}^\frown \langle y_n, y_{n+1}, ..., y_1 \rangle)$

where $\mathcal{Q}$ is $\exists$ if n is odd and $\forall$ if n is even.

We usually write $\models^n \varphi(\bar{a})$ instead of $\models^n (\varphi(\bar{x}), \bar{a})$.

So, for every Σ_n formula $\psi(\bar{x})$, the following is provable in ZFC:

$$\forall \bar{a}(\varphi(\bar{a}) \leftrightarrow \models^n \varphi(\bar{a})).$$

In particular, for every natural number n that codes a Σ_n sentence ψ we have that n codes a theorem of ZFC iff $ZFC \vdash \models^n \psi$.

12. Elementary substructures and the Löwenheim-Skolem Theorem

We will sometimes consider the language of set theory enriched with additional relation, function, or constant symbols, as well as the corresponding structures for these languages. E.g., structures of the form $\langle M, \in, A, a\rangle$, where A is a subset of M and $a \in M$.

Given any two structures $M \subseteq N$ in a given language, we write $M \prec_n N$ if M is a Σ_n-*elementary substructure* of N, i.e., for every Σ_n formula $\varphi(x_0, \dots, x_k)$ and every $a_0, \dots, a_k \in M$,

$$M \models \varphi(a_0, \dots, a_k) \text{ if and only if } N \models \varphi(a_0, \dots, a_k).$$

M is an *elementary substructure* of N, written $M \prec N$, if $M \prec_n N$ for all n. Thus, if $M \prec N$, M and N satisfy the same sentences.

The *Löwenheim-Skolem Theorem* for first-order logic asserts that for every infinite cardinal κ, every structure M for a countable language, and every $X \subseteq M$ of cardinality κ, there is an elementary substructure N of M with $X \subseteq N$ and such that N has cardinality κ. In particular, every infinite structure M for a countable language has a countable elementary substructure. The structure N, called the *Skolem Hull of* X is obtained by closing X under a family of *Skolem functions*, one for each existential formula. More precisely, for each existential formula $\exists x\varphi(x, y_1, \dots, y_n)$, one has a function $f : M^n \to M$ that assigns to each n-tuple $\langle a_1, \dots, a_n\rangle$ a witness to the sentence $\exists x\varphi(x, a_1, \dots, a_n)$, whenever the sentence holds in M, and some fixed element of M otherwise. Every well-ordering of M gives rise to a family of definable Skolem functions, namely, $f(a_1, \dots, a_n)$ is defined as *the least* witness to $\exists x\varphi(x, a_1, \dots, a_n)$ under the well-ordering.

Lecture I

13. The Reflection Theorem

For every natural number n, the following is a theorem of ZFC.

THEOREM 13.1. *There is a club class C_n of cardinals such that for every $\kappa \in C_n$,*

$$V_\kappa \prec_n V$$

i.e., for all $\kappa \in C_n$, all $\bar{a} \in V_\kappa$ and all $\varphi(\bar{x}) \in \Sigma_n$,

$$Sat(V_\kappa, \varphi(\bar{x}), \bar{a}) \quad \textit{iff} \quad \models^n \varphi(\bar{a})$$

PROOF. For $n = 0$ this is clear, since we may take C_0 to be the class of all cardinals.

Suppose now we have proved the Theorem for n, and so we have C_n.

Given $\alpha \in C_n$, let $f(\alpha) \in C_n$ be the least cardinal such that for every formula $\exists x \varphi(x, x_1, ..., x_k)$, where φ is Π_n, and every $a_1, ..., a_k$ in V_α, if $\exists x \varphi(x, a_1, ..., a_k)$, then $\varphi(b, a_1, ..., a_k)$ for some $b \in V_{f(\alpha)}$. For each $n < \omega$, let $f^n(\alpha)$ be the n-iterate of f at α. Let $F(\alpha)$ be the limit of all $f^n(\alpha)$, $n < \omega$. Note that since f is continuous, $F(\alpha)$ is a cardinal. Then $C_{n+1} = \{F(\alpha) : \alpha \in C_n\}$ is as required. $\square$

Notice that for every ordinal α we have $V_\alpha \prec_0 V$.

EXERCISE 13.2. *If $V_\alpha \prec_1 V$, then α must be an uncountable cardinal.*

One may naturally wonder whether there can be a *regular* cardinal κ such that $V_\kappa \prec_1 V$. This leads us to the first of the large cardinals.

14. Inaccessible cardinals

A cardinal κ is called *(strongly) inaccessible* if it is uncountable, regular, and a strong limit, i.e., for every cardinal $\lambda < \kappa$, $2^\lambda < \kappa$.

If κ is inaccessible, then $|V_\kappa| = \kappa$ and $\kappa = \aleph_\kappa$.

We shall see next that κ is inaccessible if and only if it is regular and V_κ is a model of ZFC. It follows, by Gödel's Second Incompleteness Theorem, that one cannot prove in ZFC that inaccessible cardinals exist.

THEOREM 14.1. *The following are equivalent for a regular cardinal κ:*

(1) *κ is inaccessible.*
(2) *$V_\kappa \models ZFC$.*

(3) $V_\kappa \prec_1 V$, *i.e.*, V_κ *is a* Σ_1*-elementary substructure of* V.

Proof. (1) implies (2): Let us check that V_κ satisfies Replacement. So, suppose F is a class function in V_κ whose domain is an element of V_κ. Thus, F has cardinality less than κ, and since κ is regular, F is not cofinal in V_κ and so it is contained in some V_α, $\alpha < \kappa$. But then the range of F belongs to $V_{\alpha+1}$.

(2) implies (3): Let $\exists x\psi(x, a)$ be a Σ_1 sentence, with parameter $a \in V_\kappa$, and suppose $\exists x\psi(x, a)$ holds. Notice that since $V_\kappa \models ZFC$, $|TC(a)| < \kappa$. Let b be a witness to $\exists x\psi(x, a)$ and let λ be a regular cardinal greater than κ such that $b \in V_\lambda$. Let N be an elementary substructure of V_λ with $b \in N$, $TC(\{a\}) \subseteq N$ and has cardinality $< \kappa$. Let M be the Mostowski collapse of N. Let c be the collapse of b. Since a collapses to itself, $M \models \psi(c, a)$. Hence, since Σ_1 sentences are upwards-absolute for transitive models, $V_\kappa \models \exists x\psi(x, a)$.

(3) implies (1): We check that κ is strong limit. So, suppose λ is a cardinal less than κ. Then, $\exists\alpha\exists f(f : \alpha \to V_{\lambda+1}$ is onto). But this is a Σ_1 sentence with $V_{\lambda+1}$ as a parameter and so it holds in V_κ. □

Theorem 14.2 (Levy, 1960). *A cardinal* κ *is inaccessible if and only if for every* $A \subseteq V_\kappa$ *there is a* $\lambda < \kappa$ *(equivalently, a club set of* λ*s) such that*

$$\langle V_\lambda, \in, A \cap V_\lambda\rangle \prec \langle V_\kappa, \in, A\rangle.$$

Proof. Suppose κ is inaccessible and let $A \subseteq V_\kappa$. Build a chain of elementary substructures of $\langle V_\kappa, \in, A\rangle$, each structure in the chain of size $< \kappa$, so that the union of the chain is of the form $\langle V_\lambda, \in, A \cap V_\lambda\rangle$, some $\lambda < \kappa$.

For the other direction, suppose κ is singular. Let A be a function whose domain is some $\mu < \kappa$ and whose range is cofinal on κ. Let $\lambda > \mu$ be such that $\langle V_\lambda, \in, A \cap V_\lambda\rangle \prec \langle V_\kappa, \in, A\rangle$. Then, the range of A is contained in λ, which is impossible. That κ is a strong limit is shown by a similar argument. □

15. Mahlo cardinals

If κ is inaccessible, then the set C of all strong limit cardinals smaller than κ is club (Exercise). So if κ is the least inaccessible cardinal, then all cardinals in C must be singular, for otherwise there would be an inaccessible cardinal below κ.

An inaccessible cardinal κ is called *Mahlo* (after the German mathematician Paul Mahlo, 1883-1971) if the set of inaccessible cardinals smaller than κ is stationary. Thus κ is Mahlo if and only if it is inaccessible and every club subset of κ contains an inaccessible cardinal. Therefore the first Mahlo cardinal, if it exists, is much greater than the first inaccessible cardinal.

One cannot prove from ZFC plus the existence of an inaccessible cardinal that a Mahlo cardinal exists. For suppose $\kappa < \lambda$ are the first two inaccessible cardinals. Then V_λ is a model of ZFC which satisfies "There exists an inaccessible cardinal" plus "There is no Mahlo cardinal".

EXERCISE 15.1. *Show that if κ is Mahlo, then the set of inaccessible cardinals smaller than κ that are themselves limits of inaccessible cardinals is stationary.*

THEOREM 15.2 (Levy, 1960). *A cardinal κ is Mahlo if and only if for every $A \subseteq V_\kappa$ there is a regular (equivalently, an inaccessible) cardinal $\lambda < \kappa$ (equivalently, a stationary set of λs) such that*

$$\langle V_\lambda, \in, A \cap V_\lambda \rangle \prec \langle V_\kappa, \in, A \rangle.$$

PROOF. Similarly as in 14.2. For the if direction, suppose C is a club subset of κ. Let $\lambda < \kappa$, λ inaccessible, be such that

$$\langle V_\lambda, \in, C \cap V_\lambda \rangle \prec \langle V_\kappa, \in, C \rangle.$$

Then C is unbounded in λ. Hence, $\lambda \in C$. $\square$

Thus, if κ is Mahlo, then for every n there is a club $C \subset \kappa$ such that $V_\alpha \prec_n V_\kappa$, for all $\alpha \in C$.

16. Indescribable and weakly-compact cardinals

Pushing the reflection principles a bit further, we can ask: Why should we restrict to first-order logic?

In second-order logic we have two kinds of variables: first-order variables $x, y, z, \ldots$, and second-order variables $X, Y, Z \ldots$, which may also be quantified. We also have predicates $X(x)$. Second-order variables are interpreted in a given structure $\langle M, \ldots \rangle$ as subsets of M, and the predicates $X(x)$ are interpreted as $x \in X$.

A second order formula is called Σ^1_0 (or Π^1_0) if its quantifiers range only over variables of first order, but it may have free variables of second order.

A formula is Σ^1_1 if it is of the form

$$\exists X_0, \ldots, X_k \varphi(X_0, \ldots, X_k, Y_0, \ldots, Y_l)$$

where $\varphi(X_0, \ldots, X_k, Y_0, \ldots, Y_l)$ is Σ^1_0.

A formula is Π^1_1 if it is of the form

$$\forall X_0, \ldots, X_k \varphi(X_0, \ldots, X_k, Y_0, \ldots, Y_l)$$

where $\varphi(X_0, \ldots, X_k, Y_0, \ldots, Y_l)$ is Σ^1_0.

Notice that, by Proposition 14.2, κ is inaccessible iff for every $A \subseteq V_\kappa$ and every Σ^1_0 sentence φ in the language of set theory with one additional predicate symbol for A, if $\langle V_\kappa, \in, A \rangle \models \varphi$, then for some $\lambda < \kappa$, $\langle V_\lambda, \in, A \cap V_\lambda \rangle \models \varphi$.

We say that κ is Σ^1_1-*indescribable* (Π^1_1-*indescribable*) if for every $A \subseteq V_\kappa$ and every Σ^1_1 (Π^1_1) sentence φ in the language of set theory with one additional predicate symbol for A, if $\langle V_\kappa, \in, A \rangle \models \varphi$, then there is $\lambda < \kappa$ such that $\langle V_\lambda, \in, A \cap V_\lambda \rangle \models \varphi$.

We have the following characterization of inaccessibility.

EXERCISE 16.1. *κ is Σ_1^1-indescribable iff it is inaccessible.*

However, Π_1^1-indescribability leads to the next large-cardinal notion.

16.1. Weakly-compact cardinals. Weakly-compact cardinals were studied by Paul Erdös and Alfred Tarski in the context of the partition calculus. Namely, κ is *weakly-compact* if κ is an uncountable cardinal and satisfies $\kappa \to (\kappa)^2$, i.e., for every coloring of all pairs of elements of κ with two colors, there is a subset X of κ of cardinality κ such that every pair of elements of X has the same color. Thus, weak-compactness generalizes Ramsey's theorem to the uncountable.

We shall give next a characterization of weakly-compact cardinals in terms of trees.

Recall that a *tree* $T = \langle T, \leq\rangle$ is a partially ordered set (poset) such that for every $t \in T$, the set $\{s \in T : s < t\}$ is well-ordered by $\leq$.

The elements of T are usually called *nodes*.

The level α of T consists of all nodes t such that the set $\{s \in T : s < t\}$ has order-type α.

The *height* of T is the least ordinal α such that the α-th level of T is empty.

A *branch* in T is a maximal linearly-ordered subset of T.

PROPOSITION 16.2 (König's Lemma. D. König, 1927). *Every infinite tree whose levels are all finite has an infinite branch.*

PROOF. Pick t_0 in level 0 such that the set $\{s : t_0 < s\}$ is infinite. Such a t_0 exists, for otherwise the level 0 would be infinite. Given t_n in level n such that the set $\{s : t_n < s\}$ is infinite, pick t_{n+1} in level $n+1$ such that $t_n \leq t_{n+1}$ and such that $\{s : t_{n+1} < s\}$ is infinite. Again, this is possible because otherwise the $n+1$-th level would be infinite. And so on. Then the set $\{t_n : n < \omega\}$ is linearly ordered and infinite, hence contained in an infinite branch. □

Is the same true for uncountable trees? That is, is it true that every uncountable tree whose levels are all countable has an uncountable branch?

An *Aronszajn κ-tree* is a tree of height κ with levels of size $< \kappa$ and with no branch of size κ.

EXERCISE 16.3. *Show that if κ is a singular cardinal, then there is an Aronszajn κ-tree. (Hint: Let $\{\alpha_\xi : \xi < \lambda\}$, $\lambda < \kappa$, be a sequence cofinal on κ and consider the tree consisting on the disjoint union of the α_ξ, $\xi < \lambda$, each with the ordinal ordering.)*

LEMMA 16.4. *If κ is weakly-compact, then κ is inaccessible.*

PROOF. Suppose $\kappa = \bigcup\{X_\alpha : \alpha < \lambda\}$, where all the X_α are pairwise disjoint, $\lambda < \kappa$ and $|X_\alpha| < \kappa$, all $\alpha < \lambda$. Let f be the coloring given by: $f(\{\beta, \gamma\}) = 1$ iff β and γ belong to the same X_α. Then f has no homogeneous set of size κ. This shows κ is regular.

To see that κ is a strong limit, suppose, towards a contradiction, that $\{g_\alpha : \alpha < \kappa\}$ is a collection of functions from a fixed $\lambda < \kappa$ into 2. Let f be the coloring given by: $f(\{\alpha, \beta\}) = 1$ iff $g_\alpha <_{lex} g_\beta$ iff $\alpha < \beta$, i.e. the lexicographic ordering agrees with the ordering of the subindices. An f-homogeneous set produces an increasing or a decreasing sequence under the lexicographic ordering. But it is a general fact that there cannot be any such sequence of length λ^+: for suppose $\{h_\alpha : \alpha < \lambda^+\}$ is an increasing sequence. Let $\gamma \leq \lambda$ be the least ordinal such that $\{h_\alpha \restriction \gamma : \alpha < \lambda^+\}$ has size λ^+. So, we may assume all the $h_\alpha \restriction \gamma$ are distinct. For each α, let δ_α be the least ordinal where h_α and $h_{\alpha+1}$ differ. Note that $\delta_\alpha < \gamma$. So, we may assume all δ_α are the same, call it δ. But if $h_\alpha \restriction \delta = h_\beta \restriction \delta$, then $h_\beta <_{lex} h_{\alpha+1}$ and $h_\alpha <_{lex} h_{\beta+1}$. Hence, $\alpha = \beta$. Thus, $\{h_\alpha \restriction \delta : \alpha < \lambda^+\}$ has size λ^+, contradicting the minimality of γ. □

The following is a useful characterization of weakly-compact cardinals.

THEOREM 16.5. *κ is weakly-compact iff it is inaccessible and there are no Aronszajn κ-trees.*

PROOF. Suppose T is a tree of height κ with all levels of size $< \kappa$. We may assume that T is a tree on κ. Extend $<_T$ to a linear-ordering $\prec$ as follows: if $\alpha <_T \beta$, then $\alpha \prec \beta$, and if α and β are incomparable, then let $\alpha \prec \beta$ iff in the first level where they split, their predecessors at that level are $<$-ordered in the same way. i.e., if γ is the first level of T where the branches leading to α and β split, and if α_0 and β_0 are the predecessors of α and β, respectively, at level γ, then $\alpha_0 < \beta_0$. We Define $F : [\kappa]^2 \to 2$ by $F(\{\alpha, \beta\}) = 1$ iff $\prec$ agrees with $<$ on $\{\alpha, \beta\}$. By weak compactness let $H \subseteq \kappa$ be homogeneous for F and of size κ. Consider the set B of all α such that there are κ-many elements of H above α in the tree ordering. Then B is a chain: For suppose $\alpha, \beta \in B$ are such that $\alpha \prec \beta$, $\alpha \not<_T \beta$ and $\beta \not<_T \alpha$. Pick $\alpha' < \beta' < \gamma$ in H such that $\alpha <_T \alpha', \gamma$ and $\beta <_T \beta'$. Then, $F(\{\alpha', \beta'\}) = 1$, but $F(\{\beta', \gamma\}) = 0$.

Now suppose κ is inaccessible and let $F : [\kappa]^2 \to 2$.

We construct the nodes of a tree T: let $t_0 = \emptyset$. Suppose $\{t_\beta : \beta < \alpha\}$ have already been constructed, where each t_β is a function from some $\gamma \leq \beta$ into 2. We construct t_α by induction on $\gamma < \alpha$. Suppose $t_\alpha \restriction \gamma$ has already been constructed. If $t_\alpha \restriction \gamma$ is not in $\{t_\beta : \beta < \alpha\}$, then let $t_\alpha = t_\alpha \restriction \gamma$. Otherwise, $t_\alpha \restriction \gamma = t_\beta$, some $\beta < \alpha$. Then, let $t_\alpha(\gamma) = F(\{\beta, \alpha\})$.

T is a tree of height κ and, since κ is inaccessible, all levels are of size $< \kappa$. Hence, it has a chain B of size κ. For each $i \in \{0, 1\}$, let $H_i = \{\alpha : t_\alpha \in B \text{ and } t_\alpha^\frown i \in B\}$. Each H_i is homogeneous for F, hence one of them must have size κ. □

The last argument of the proof above can be easily adapted to show that if κ is inaccessible and there are no Aronszajn κ-trees, then $\kappa \to (\kappa)^n$, for every $n < \omega$. Hence, $\kappa \to (\kappa)^2$ iff $\kappa \to (\kappa)^n$, for every $n < \omega$.

The following equivalence is now surprising, for it shows that two apparently unrelated notions: a reflection principle and a partition property, are in fact equivalent. It also gives a characterization of weakly-compact cardinals is terms of elementary embeddings.

THEOREM 16.6 (Hanf and Scott 1961; Keisler 1962). *The following are equivalent for a cardinal* κ*:*

(1) κ *is* Π^1_1*-indescribable.*
(2) κ *is weakly-compact.*
(3) *For every* $A \subseteq V_\kappa$*, there is a transitive set* M *with* $\kappa \in M$ *and* $X \subseteq M$ *such that* $\langle V_\kappa, \in, A\rangle \prec \langle M, \in, X\rangle$*.*

PROOF. (1) implies (2): By Theorem 14.2, every Π^1_1-indescribable cardinal is inaccessible. So it will be enough to show that there are no κ-Aronszajn trees. Towards a contradiction, suppose T is a κ-tree on κ. For every limit $\alpha < \kappa$, $\langle V_\alpha, \in, T \cap V_\alpha\rangle$ satisfies the Σ^1_1 sentence that says "There is a branch of T of unbounded length". Hence, $\langle V_\kappa, \in, T\rangle$ satisfies the same sentence.

(2) implies (3): Fix $A \subseteq V_\kappa$. By 14.2, $C = \{\alpha < \kappa : \langle V_\alpha, \in A \cap V_\alpha\rangle \prec \langle V_\kappa, \in A\rangle\}$ is a club.

Fix a well-ordering of V_κ, so that whenever we take the Skolem hull of some $X \subseteq V_\kappa$ in $\langle V_\kappa, \in, A\rangle$ we do it with respect to this fixed well-ordering.

For every $\alpha \in C$ and every β with $\alpha < \beta < \kappa$, let $H(\alpha, \beta)$ be the Skolem hull of $V_\alpha \cup \{\beta\}$ in $\langle V_\kappa, \in, A\rangle$.

Let $H(\alpha, \beta) \sim H(\alpha', \beta')$ iff $\alpha = \alpha'$ and $H(\alpha, \beta)$ and $H(\alpha', \beta')$ are isomorphic, via an isomorphism that is the identity on V_α and sends β to β'. It is clear that $\sim$ is an equivalence relation. Note that, by inaccessibility of κ, for each $\alpha \in C$ there is β such that $[H(\alpha, \beta)]$ has cardinality κ.

Let T be the set of all $\sim$-equivalence classes of cardinality κ ordered by: $[H(\alpha, \beta)] <_T [H(\alpha', \beta')]$ iff $\alpha < \alpha'$, $\beta \leq \beta'$ and the map $j : V_\alpha \cup \{\beta\} \to V_{\alpha'} \cup \{\beta'\}$ that is the identity on V_α and sends β to β' extends to an elementary embedding $j : H(\alpha, \beta) \to H(\alpha', \beta')$. We claim that $\langle T, <_T\rangle$ is a tree.

$<_T$ is clearly well-founded. To see that below any node $<_T$ is a linear ordering, suppose $[H(\alpha, \beta)]$, $[H(\alpha', \beta')] <_T [H(\alpha'', \beta'')]$, where $\alpha \leq \alpha'$. Since each equivalence class has cardinality κ, we may assume $\beta \leq \beta'$. Let $j : H(\alpha, \beta) \to H(\alpha'', \beta'')$ and $j' : H(\alpha', \beta') \to H(\alpha'', \beta'')$ be the corresponding elementay embeddings. Then there is a unique map $k : H(\alpha', \beta') \to H(\alpha'', \beta'')$ such that $j' \circ k = j$, witnessing $[H(\alpha, \beta)] <_T [H(\alpha', \beta')]$.

Since κ is inaccessible, T is a κ-tree. Thus, by weak compactness (Theorem 16.5), let $\langle [H(\alpha, \beta_\alpha)] : \alpha < \kappa\rangle$ be a branch through T. So, if $\alpha \leq \alpha' < \kappa$, we have an elementary embedding $i_{\alpha,\alpha'} : H(\alpha, \beta_\alpha) \to H(\alpha', \beta_{\alpha'})$ that fixes V_α and sends β_α to $\beta_{\alpha'}$. Moreover, if $\alpha \leq \alpha' \leq \alpha'' < \kappa$, then $i_{\alpha,\alpha''} = i_{\alpha',\alpha''} \circ i_{\alpha,\alpha'}$. Let $N = \langle N, E, Y\rangle$ be the direct limit of $\langle H(\alpha, \beta_\alpha) : \alpha < \kappa\rangle$. Since κ has uncountable cofinality, N is well-founded. Let $\langle M, \in, X\rangle$ be the transitive collapse of N. Then,

$\langle V_\kappa, \in, A\rangle \prec \langle M, \in, X\rangle$. Moreover, since $[\langle \alpha, \beta_\alpha\rangle] = [\langle \alpha', \beta_{\alpha'}\rangle]$, for all $\alpha, \alpha' < \kappa$, the transitive collapse of $[\langle \alpha, \beta_\alpha\rangle]$ is $\geq \kappa$, and so $\kappa \in M$.

(3) implies (1): Let $A \subseteq V_\kappa$ and suppose $\langle V_\kappa, \in, A\rangle \models \forall Z\varphi(Z)$, where $\forall Z\varphi(Z)$ is a Π^1_1 sentence, with $\varphi(Z)$ being first-order with Z as a second-order variable predicate and which may have A as a parameter predicate. By (3), let $\langle M, \in, X\rangle$, with M transitive and $\kappa \in M$ be such that $\langle V_\kappa, \in, A\rangle \prec \langle M, \in, X\rangle$. Note that $V^M_\kappa = V_\kappa$ and so $V_\kappa \in M$. Moreover, $A = X \cap V_\kappa$. Since $\forall Z\varphi(Z)$ is Π^1_1,

$$\langle M, \in, X\rangle \models \text{“}\langle V_\kappa, \in, A\rangle \models \forall Z\varphi(Z)\text{”}.$$

Hence,

$$\langle M, \in, X\rangle \models \exists\alpha(\langle V_\alpha, \in, X \cap V_\alpha\rangle \models \forall Z\varphi(Z)).$$

But the right-hand side is a first-order sentence, hence by elementarity,

$$\langle V_\kappa, \in, A\rangle \models \exists\alpha(\langle V_\alpha, \in, A \cap V_\alpha\rangle \models \forall Z\varphi(Z)).$$

Therefore, there is $\alpha < \kappa$ such that

$$\langle V_\alpha, \in, A \cap V_\alpha\rangle \models \forall Z\varphi(Z).$$

□

16.2. Stationary reflection.

THEOREM 16.7 (Stationary Reflection). *If κ is weakly compact, then for every collection $\{S_\alpha : \alpha < \kappa\}$ of stationary subsets of κ, there exists an inaccessible λ such that $S_\alpha \cap \lambda$ is stationary, for all $\alpha < \lambda$.*

PROOF. Let $A = \{\langle \alpha, \beta\rangle : \beta \in S_\alpha\}$. Let $F : V_\kappa \to \kappa$ be such that if λ is a cardinal, then $F(\lambda) = 2^\lambda$, and if f is a function from some ordinal α into κ, then $F(f) = sup(range(f))$. Such an F exists because κ is inaccessible.

The sentence: “Every S_α, $\alpha < \kappa$, is stationary” can be expressed as a Π^1_1 sentence over $\langle V_\kappa, \in, A, F\rangle$. Indeed,

$$\forall C\forall\alpha(C \text{ is club} \to \exists\beta \in C(\langle \alpha, \beta\rangle \in A)).$$

And the sentence : “For every function $f : \alpha \to \kappa$, $F(f)$ exists” can also be expressed as a Π^1_1 sentence over $\langle V_\kappa, \in, A, F\rangle$. Namely,

$$\forall f(\exists\alpha(\alpha \in OR \wedge dom(f) = \alpha) \wedge range(f) \subseteq OR \to \exists\beta F(f) = \beta).$$

Since κ is Π^1_1-indescribable, there exists $\lambda < \kappa$ such that $\langle V_\lambda, \in A \cap V_\lambda, F \cap V_\lambda\rangle$ satisfies

$$\forall C\forall\alpha(C \text{ is club} \to \exists\beta \in C(\langle \alpha, \beta\rangle \in A \cap V_\lambda))$$

and also

$$\forall f(\exists\alpha(\alpha \in OR \wedge dom(f) = \alpha) \wedge range(f) \subseteq OR \to \exists\beta F \cap V_\lambda(f) = \beta).$$

The first sentence implies that $S_\alpha \cap \lambda$ is stationary in λ, for every $\alpha < \lambda$. And the second sentence that λ is regular. Finally, since V_λ is closed under F, λ must be a strong limit cardinal. □

COROLLARY 16.8. *Every weakly-compact cardinal is Mahlo.*

PROOF. Let R be the set of regular cardinals below κ. Let C be a club subset of κ. By 16.7, let $\lambda \in R$ be such that $C \cap \lambda$ is stationary in λ. Then, $\lambda \in C$. $\square$

Lecture II

17. Erdös cardinals

Another possible strengthening of $\kappa \to (\kappa)^2$, or rather its equivalent form: for every $n < \omega$, $\kappa \to (\kappa)^n$, would be to require the existence of sets that are simultaneously homogeneous for all $n < \omega$. Namely, for X a set, let $[X]^{<\omega}$ be the set of all finite subsets of X. For α an ordinal and κ a cardinal, the notation $\kappa \to (\alpha)^{<\omega}$ means that for every coloring of $[\kappa]^{<\omega}$ into two colors, there is a homogeneous set of order-type α, i.e., a subset X of κ of order-type α such that for every n, all elements of $[X]^n$ have the same color. Notice that we cannot require that all elements of $[X]^{<\omega}$ have the same color, since, e.g., we could color $[\kappa]^1$ all green and $[\kappa]^2$ all red.

If $\alpha \geq \omega$, the *α-Erdös cardinal* is the least cardinal κ such that $\kappa \to (\alpha)^{<\omega}$. We denote such a κ, if it exists, by $\kappa(\alpha)$.

Erdös cardinals can be characterized in terms of *indiscernibles.* Namely,

Lemma 17.1 (J. H. Silver). *For $\alpha \geq \omega$, we have $\kappa \to (\alpha)^{<\omega}$ iff for every structure M in a countable language with $\kappa \subseteq M$, there is a set $X \subseteq \kappa$ of order-type α of M-indiscernibles. i.e., for every formula $\varphi(x_1, ..., x_n)$ in the language of M, and every $\alpha_1 < ... < \alpha_n$ and $\beta_1 < ... < \beta_n$ in X,*

$$M \models \varphi(\alpha_1, ..., \alpha_n) \text{ iff } M \models \varphi(\beta_1, ..., \beta_n).$$

Proof. Let $\{\varphi_n : n < \omega\}$ be an enumeration of all the formulas of the language of M so that φ_n has at most n free variables. Let $f : [\kappa]^{<\omega} \to 2$ be given by: $f(\alpha_1, ..., \alpha_n) = 0$ iff $M \models \varphi_n(\alpha_1, ..., \alpha_{i_n})$. Then any f-homogeneous set of order-type α is a set of M-indiscernibles.

Conversely, if $f : [\kappa]^{<\omega} \to 2$ and X is a set of indiscernibles for the structure $\langle \kappa, \in, f \restriction [\kappa]^n \rangle_{n\in\omega}$, then X is f-homogeneous. □

How large are Erdös cardinals? It is not very hard to see that $\kappa(\omega)$ is Π^1_1-describable and so it is not weakly-compact. It can be shown, however, that $\kappa(\omega)$ is inaccessible. Even though $\kappa(\omega)$ itself has not very strong large-cardinal properties, there are very large cardinals below it.

Theorem 17.2 (Reinhardt and Silver). *There is a totally indescribable cardinal below $\kappa(\omega)$.*

Proof. Let $\kappa = \kappa(\omega)$. Let W be a well-ordering of V_κ and I a set of ω indiscernibles for $\langle V_\kappa, \in W\rangle$. Let $N \prec V_\kappa$ be the Skolem hull of I in V_κ with respect to Skolem functions defined with W. Let $\bar{N}$ be the transitive collapse of N and let π be the inverse collapsing isomorphism. Since κ is inaccessible, $\bar{N} \models ZFC$. Let $f : I \to I$ be any order-preserving injection which is not the identity. f induces an elementary embedding $j : \bar{N} \to \bar{N}$ which is not the identity. Let λ be the critical point of the embedding. It will be enough to show that $\bar{N} \models$ "λ is totally indescribable", for then $\pi(\lambda)$ is totally indescribable in N, hence in V_κ.

So, suppose φ is Π^m_n, some m, n. Suppose that

$$\bar{N} \models (A \subseteq V_\lambda \wedge \langle V_\lambda, \in, A\rangle \models \varphi)$$

Then,

$$\bar{N} \models \exists \alpha < j(\lambda)(\langle V_\alpha, \in, j(A) \cap V_\alpha\rangle \models \varphi)$$

By elementarity,

$$\bar{N} \models \exists \alpha < \lambda(\langle V_\alpha, \in, A \cap V_\alpha\rangle \models \varphi)$$

□

PROPOSITION 17.3. *If $\kappa(\omega)$ exists, then $L \models$ "$\kappa(\omega)$ exists".*

PROOF. Let $\kappa = \kappa(\omega)$. It is enough to show that κ satisfies $\kappa \to (\omega)^{<\omega}$ in L. So, suppose $f : [\kappa]^{<\omega} \to 2$ belongs to L. Let T be the tree of finite f-homogeneous increasing sequences. Clearly, $T \in L$. We have that f has an infinite homogeneous set iff T is ill-founded. Now, T is ill-founded in V. Hence, by absoluteness of ill-foundedness, T is ill-founded in L. □

EXERCISE 17.4. *Prove the proposition above for every ordinal $\alpha < \omega_1^L$.*

Let us look now at $\kappa(\omega_1^L)$. The following consequence of the existence of $\kappa(\omega_1^L)$ follows from work of Rowbottom and Silver and shows, in particular, that the ω_1^L-Erdös cardinal does not exist in L, and therefore its existence implies $V \neq L$.

THEOREM 17.5. *If $\kappa(\omega_1^L)$ exists, then in L there are only countably-many subsets of ω.*

PROOF. Let $\kappa = \kappa(\omega_1^L)$. Since $|L_\kappa| = \kappa$, let I be a set of indiscernibles for L_κ of order-type ω_1^L. Let M be the Skolem hull of I in L_κ constructed using the definable canonical well-ordering of L_κ. Let N be the transitive collapse of M. Thus $N = L_\lambda$, for some $\lambda \geq \omega_1^L$, and so every constructible subset of ω is in N. Let J be the image of I under the transitive collapse.

Every $x \in N$ is the least (in the canonical well-ordering of N) such that $\varphi(x, \alpha_1, ..., \alpha_n)$, for some formula φ and some increasing sequence $\alpha_1 < ... < \alpha_n$ in J. Let $\langle \beta_n : n < \omega\rangle$ be an increasing enumeration of the first ω elements of J. Since each $m \in \omega$ is definable, if x is a subset of ω so that x is the least such that $\varphi(x, \alpha_1, ..., \alpha_n)$, then by indiscernibility, for every $m \in \omega$, m belongs to the least x such that $\varphi(x, \alpha_1, ..., \alpha_n)$ iff m belongs to the least x such that $\varphi(x, \beta_1, ..., \beta_n)$. Thus, every subset of ω that belongs to N is determined by a formula and a finite initial sequence of the first ω indiscernibles of J. □

EXERCISE 17.6. *Prove the theorem above relativized to a subset of* ω*. Namely, if* $a \subseteq \omega$ *and* $\kappa(\omega_1^{L[a]})$ *exists, then there are only countably-many subsets of* ω *in* $L[a]$*.*

COROLLARY 17.7. *If* $\kappa(\omega_1)$ *exists, then* ω_1 *is an inaccessible cardinal in* $L[a]$*, for every* $a \subseteq \omega$*.*

PROOF. Fix $a \subseteq \omega$. We only need to show that ω_1 is a limit cardinal in $L[a]$. So, towards a contradiction, suppose that λ is a cardinal in L and $(\lambda^+)^{L[a]} = \omega_1$. Since λ is countable, let $b \subseteq \omega$ code a well-ordering of ω of order-type λ, so that in $L[a, b]$, λ is countable. Then $\omega_1^{L[a,b]} = \omega_1$, which contradicts the last exercise. □

18. $0^\sharp$

Notice that to obtain the conclusion of Theorem 17.5 and the Corollary above we only need that for some limit ordinal λ, L_λ has a set I of indiscernibles of order-type ω_1^L.

Equivalently, by taking the canonical Skolem hull of I in L_λ and then taking the transitive collapse, we may also require that I generates L_λ, i.e., every element of L_λ is definable in L_λ from a finite set of indiscernibles (see the proof of Theorem 17.5). Then any order-preserving injection $f : I \to I$ induces an elementary embedding $j : L_\lambda \to L_\lambda$.

It follows that if $\kappa \in I$ is a cardinal in L_λ, then κ is totally indescribable in L_λ.

Suppose now that there is a limit ordinal λ such that L_λ has an uncountable set I of indiscernibles, a hypothesis that follows from the existence of $\kappa(\omega_1)$ by the same arguments as before. This hypothesis, weaker than the existence of $\kappa(\omega_1)$, is known as $0^\sharp$ *exists.*

It follows from work of Silver and Kunen that the existence of $0^\sharp$ is actually equivalent to the existence of a non-trivial elementary embedding $j : L \to L$.

Furthermore, Silver showed that the existence of $0^\sharp$ implies that there is a club class of indiscernibles for L, which generate L, and which contains all uncountable cardinals in V. Hence, all such cardinals are totally indescribable in L.

19. Ultrafilters

DEFINITION 19.1. *A filter* $\mathcal{F}$ *on a set* A *is called an* ultrafilter *if for every* $X \subseteq A$*, either* $X \in \mathcal{F}$ *or* $A - X \in \mathcal{F}$*.*

A filter $\mathcal{F}$ on A is called *maximal* if there is no filter on A that properly contains $\mathcal{F}$. i.e., if for every filter $\mathcal{G}$ on A, if $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{F} = \mathcal{G}$.

PROPOSITION 19.2. *A filter* $\mathcal{F}$ *on* A *is maximal if and only if it is an ultrafilter.*

Proof. If $\mathcal{F}$ is an ultrafilter, then it is clearly maximal, for the addition of any new $X \subseteq A$ to $\mathcal{F}$ would imply that X and its complement are both in $\mathcal{F}$, and then $X \cap (A - X) = \emptyset \in \mathcal{F}$.

Now suppose $\mathcal{F}$ is a maximal filter and $X \subseteq A$. Suppose that neither X nor its complement belong to $\mathcal{F}$. Then for every $Y \in \mathcal{F}$ we have $X \cap Y \neq \emptyset$, for otherwise $Y \subseteq (A - X)$ and therefore $A - X \in \mathcal{F}$. It follows that $\mathcal{F} \cup \{X\}$ has the finite intersection property, hence it can be extended to a filter $\mathcal{G}$. But since $X \in \mathcal{G} - \mathcal{F}$, $\mathcal{F}$ is not maximal. A contradiction. □

Theorem 19.3 (A. Tarski). *Every filter can be extended to an ultrafilter.*

Proof. Let $\mathcal{F}$ be a filter on some set A. Let $\mathbb{P}$ be the set of all filters on A that contain $\mathcal{F}$, ordered by $\subseteq$. Then $\mathbb{P}$ is a partial ordering. If C is a chain in $\mathbb{P}$, then $\bigcup C$ is also a filter on A, and therefore an upper bound of C in $\mathbb{P}$. Hence by Zorn's Lemma $\mathbb{P}$ has a maximal element which, by the Proposition above, is an ultrafilter. □

An ultrafilter $\mathcal{F}$ on a set A is called *principal* if and only if there exists $a \in A$ such that $\mathcal{F} = \{X \subseteq A : a \in X\}$.

Exercise 19.4. *Show that every filter on a finite set A is principal.*

An example of a non-principal filter on ω is the *Fréchet filter*, which is the set of all co-finite subsets of ω, i.e., $\{X \subseteq \omega : \omega - X \text{ is finite}\}$. More generally, if κ is an infinite cardinal, then the set of all subsets of κ whose complement has cardinality less than κ is a filter.

We say that a family F of subsets of a set A has the *finite-intersection property* if the intersection of any finite number of sets in F is non-empty. Clearly, every filter has the finite intersection property.

If $F \subseteq \mathcal{P}(A)$ is non-empty and has the finite intersection property, then F can be extended to a filter on A. Indeed, let $\mathcal{F}$ be the set of all subsets of A that contain some finite intersection of sets from F. Then one can easily check that $\mathcal{F}$ is a filter. (Exercise.)

19.1. κ-complete ultrafilters. Let κ be an infinite cardinal. A filter $\mathcal{F}$ on a set A is called *κ-complete* if the intersection of less than κ-many elements of $\mathcal{F}$ belongs to $\mathcal{F}$. ω_1-complete filters are also called *σ-complete*.

Note that every principal filter on a set A is κ-complete, for every κ. There is no σ-complete non-principal filter on any countable set (Exercise). The filter $\{X \subseteq \omega_1 : |\omega_1 - X| \leq \aleph_0\}$ is σ-complete. More generally, for every uncountable regular cardinal κ, the filter $\{X \subseteq \kappa : |\kappa - X| < \kappa\}$ is κ-complete. The filter of subsets of $[0, 1]$ of Lebesgue measure 1 is σ-complete.

A natural question is if there exists a σ-complete non-principal ultrafilter on some set A, equivalently on some cardinal κ.

PROPOSITION 19.5. *Suppose $\lambda \leq \kappa$ are infinite cardinals. An ultrafilter $\mathcal{F}$ on κ is λ-complete if and only if for every partition $\{X_\alpha : \alpha < \mu\}$ of κ, where $\mu < \lambda$, there exists α such that $X_\alpha \in \mathcal{F}$.*

PROOF. $\Rightarrow$. Suppose $\{X_\alpha : \alpha < \mu\}$, some $\mu < \lambda$, is a partition of κ. If none of the X_α's is in $\mathcal{F}$, then $\kappa - X_\alpha \in \mathcal{F}$, for all $\alpha < \mu$. Hence by λ-completeness, $\bigcap_{\alpha<\mu}(\kappa - X_\alpha) = \emptyset \in \mathcal{F}$, which is impossible.

$\Leftarrow$. By induction on λ. So assume $\mathcal{F}$ is λ-complete and let us show that it is λ^+-complete.

Given $\{X_\alpha : \alpha < \lambda\} \subseteq \mathcal{F}$, let $Y_0 = X_0$, let $Y_{\alpha+1} = Y_\alpha \cap X_{\alpha+1}$, and for α limit let $Y_\alpha = \bigcap_{\beta<\alpha} Y_\beta$. By the inductive assumption, all Y_α belong to $\mathcal{F}$.

Now let $Z_\alpha = Y_\alpha - Y_{\alpha+1}$. Thus,

$$\{\kappa - X_0\} \cup \{Z_\alpha : \alpha < \mu\} \cup \{\bigcap_{\alpha<\mu} Y_\alpha\}$$

is a partition of κ.

Since $X_0 \in \mathcal{F}$, $\kappa - X_0 \notin \mathcal{F}$. And $Z_\alpha \notin \mathcal{F}$ for all α, because $\kappa - Z_\alpha = \kappa - (Y_\alpha - Y_{\alpha+1}) = (\kappa - Y_\alpha) \cup Y_{\alpha+1} \in \mathcal{F}$. Hence by our assumption,

$$\bigcap_{\alpha<\mu} Y_\alpha = \bigcap_{\alpha<\mu} X_\alpha \in \mathcal{F}.$$

□

EXERCISE 19.6. *Show that if $\mathcal{U}$ is a κ-complete ultrafilter on κ and $\bigcup_{\alpha<\lambda} X_\lambda \in \mathcal{U}$, where $\lambda < \kappa$, then $X_\alpha \in \mathcal{U}$ for some $\alpha < \lambda$.*

PROPOSITION 19.7. *If κ is the least cardinal for which there exists a non-principal σ-complete ultrafilter $\mathcal{F}$ on κ, then $\mathcal{F}$ is in fact κ-complete.*

PROOF. Notice that the assumption implies κ is uncountable. So, suppose, to the contrary, that $\{X_\alpha : \alpha < \lambda\}$, some infinite cardinal $\lambda < \kappa$, is a partition of κ such that $X_\alpha \notin \mathcal{F}$, for all $\alpha < \lambda$. Then define the filter $\mathcal{G}$ on λ as follows

$$X \in \mathcal{G} \text{ if and only if } \bigcup_{\alpha\in X} X_\alpha \in \mathcal{F}.$$

$\mathcal{G}$ is non-principal, for if $\alpha < \lambda$ is such that $G = \{X \subseteq \lambda : \alpha \in X\}$, then $\{\alpha\} \in G$, and therefore $X_\alpha \in \mathcal{F}$, which is impossible.

We claim that $\mathcal{G}$ is an ultrafilter, for if $X \subseteq \lambda$ is not in $\mathcal{G}$, then $\bigcup_{\alpha\in X} X_\alpha \notin \mathcal{F}$. And since $\mathcal{F}$ is an ultrafilter this implies

$$\kappa - \bigcup_{\alpha\in X} X_\alpha = \bigcap_{\alpha\in X}(\kappa - X_\alpha) = \bigcap_{\alpha\in X}\bigcup_{\beta\neq\alpha} X_\beta = \bigcup_{\alpha\in(\lambda-X)} X_\alpha \in \mathcal{F}$$

hence $\lambda - X \in \mathcal{G}$.

Suppose now that $\{Y_n : n < \omega\} \subseteq \mathcal{G}$. Then, $\bigcup_{\alpha \in Y_n} X_\alpha \in \mathcal{F}$, for every n. Since $\mathcal{F}$ is σ-complete,

$$\bigcap_{n<\omega} \bigcup_{\alpha \in Y_n} X_\alpha = \bigcup_{\alpha \in \bigcap_{n<\omega} Y_n} X_\alpha \in \mathcal{F}$$

and so $\bigcap_{n<\omega} Y_n \in \mathcal{G}$. ☐

20. Measurable cardinals

A uncountable cardinal κ is called *measurable* if there exists a κ-complete non-principal ultrafilter on κ.

By Proposition 19.7, if κ is the least cardinal on which there exists a σ-complete non-principal ultrafilter, then κ is measurable.

We say that a filter $\mathcal{F}$ on a cardinal κ is *uniform* if every $X \in \mathcal{F}$ has cardinality κ.

PROPOSITION 20.1. *Every κ-complete non-principal ultrafilter on κ is uniform.*

PROOF. Suppose $\mathcal{U}$ is a κ-complete non-principal ultrafilter on κ and assume, to the contrary that $X \in \mathcal{U}$ has cardinality λ, for some $\lambda < \kappa$. Since $\mathcal{U}$ is non-principal, for every $\alpha \in X$, there exists $X_\alpha \in \mathcal{U}$ such that $\alpha \notin X_\alpha$. Hence by κ-completeness, $Y := \bigcap_{\alpha<\lambda} X_\alpha \in \mathcal{U}$. But then $X \cap Y = \emptyset$, which is impossible. ☐

We will see that measurable cardinals are very large.

PROPOSITION 20.2. *Every measurable cardinal is inaccessible.*

PROOF. First notice that an infinite cardinal κ is regular if and only if it cannot be partitioned into less than κ-many subsets, each of size less than κ. Now suppose κ is measurable and let $\mathcal{U}$ be a κ-complete non-principal ultrafilter on κ. By Proposition 19.5 every partition of κ into less than κ-many sets, contains an element in $\mathcal{U}$, which by Proposition 20.1 must have size κ.

It only remains to show that κ is a strong limit. So suppose, to the contrary, that $2^\lambda \geq \kappa$, for some $\lambda < \kappa$. Thus, there exists a set $S = \{f_\alpha : \alpha < \kappa\}$, where $f_\alpha : \lambda \to 2$ for all $\alpha < \kappa$.

Let $\mathcal{U}$ be a κ-complete non-principal ultrafilter on κ. For each $\beta < \lambda$, let $X_\beta = \{\alpha : f_\alpha(\beta) = 0\}$. Then let ε_β be either 0 if or 1 according to whether $X_\beta \in \mathcal{U}$ or $X_\beta \notin \mathcal{U}$. Then by κ-completeness of $\mathcal{U}$, the intersection $\bigcap_{\beta<\lambda} X_\beta$ is in $\mathcal{U}$. But this intersection contains exactly one element, namely the function f such that $f(\beta) = \varepsilon_\beta$, and this is impossible because $\mathcal{U}$ is non-principal. ☐

20.1. Normal ultrafilters. A filter on a regular uncountable cardinal is called normal if it is closed under diagonal intersections. Thus, Proposition 9.4 shows that $Club(\kappa)$ is normal.

EXERCISE 20.3. *Show that, for κ regular and uncountable, the κ-complete filter $F = \{X \subseteq \kappa : |\kappa - X| < \kappa\}$ is not normal.*

Show that every principal filter on κ is normal.

EXERCISE 20.4. *Show that if $\mathcal{U}$ is a κ-complete non-principal ultrafilter on κ, then for every $\alpha < \kappa$, the tail set $C_\alpha := \{\beta < \kappa : \alpha < \beta\}$ belongs to $\mathcal{U}$.*

PROPOSITION 20.5. *If F is a normal filter on κ such that all the tail sets $C_\alpha := \{\beta < \kappa : \alpha < \beta\}$, for $\alpha < \kappa$, belong to F, then every club subset of κ belongs to F. Hence, every element of F is stationary.*

PROOF. First note that the club set D of limit ordinals smaller than κ belongs to F, because $D = \Delta_{\alpha<\kappa} C_{\alpha+1}$. Suppose now that A is club, and let $\{x_\alpha : \alpha < \kappa\}$ be its increasing enumeration. Then $D \cap \Delta_{\alpha<\kappa} C_{x_\alpha} \subseteq A$. □

PROPOSITION 20.6. *A filter F on a regular uncountable cardinal κ is normal if and only if for every regressive function f on a set $S \notin F^*$ there exists $S' \notin F^*$ contained in S on which f is constant.*

PROOF. Suppose F is normal. Then we argue as in the proof of the Pressing-Down Lemma. Suppose, towards a contradiction, that for every $\alpha < \kappa$, the set $\{\beta \in S : f(\beta) = \alpha\}$ belongs to F^*. So let $C_\alpha \subseteq \kappa$ be in F and disjoint form the set. Thus, $f(\beta) \neq \alpha$ for every $\beta \in S \cap C_\alpha$. Now let $C = \Delta_{\alpha<\kappa} C_\alpha$. Then $S \cap C \neq \emptyset$ and if $\beta \in S \cap C$, then $f(\beta) \neq \alpha$ for all $\alpha < \beta$, contradicting the fact that f is regressive on S.

For the converse, suppose $\langle X_\alpha : \alpha < \kappa \rangle$ be a sequence of sets in F. If $\Delta_{\alpha<\kappa} X_\alpha \notin F$, then the complement, call it S, does not belong to F^*. Let $f : S \to \kappa$ be so that $f(\alpha)$ is some ordinal $\beta < \alpha$ such that $\alpha \notin X_\beta$. Let $S' \notin F^*$ be contained in S and on which f is constant, say with value β. Then $X_\beta \cap S' = \emptyset$, which is impossible. □

Thus if $\mathcal{U}$ is an ultrafilter on a regular uncountable cardinal κ, then $\mathcal{U}$ is normal if and only in for every regressive function f on a set $S \in \mathcal{U}$ there exists $S' \in \mathcal{U}$ contained in S on which f is constant.

Lecture III

21. Elementary embeddings

If N and M are structures for the language of set theory, a function $j : N \to M$ is an *elementary embedding* if for every formula $\varphi(x_1, ..., x_n)$ and every $a_1, ..., a_n \in N$,

$$N \models \varphi(a_1, ..., a_n) \text{ iff } M \models \varphi(j(a_1), ..., j(a_n)).$$

Suppose now that $M \subseteq N$ are models of ZFC, with N transitive, and $j : N \to M$ is an elementary embedding which is not the identity. Then there is a least ordinal α that is moved by j. To see this, let x be a set in N of least rank such that $j(x) \neq x$. Let $\alpha = rank(x)$. Since the elements of x have rank smaller than α, $x \subseteq j(x)$. So there is $y \in j(x) \setminus x$. But then $\alpha \leq rank(y)$, since otherwise $j(y) = y \in j(x)$, and therefore by elementarity of j, $y \in x$, which is not the case. Thus, $\alpha \leq rank(y) < rank(j(x)) = j(\alpha)$.

The least ordinal α moved by j is called the *critical point of* j, denoted by $crit(j)$.

Proposition 21.1. *If $\alpha = crit(j)$, then α is an inaccessible cardinal in N.*

Proof. Let us show that α is a cardinal. Otherwise, there is $\beta < \alpha$ and a bijection $f : \beta \to \alpha$. But then, by elementarity, $j(f) : \beta \to j(\alpha)$ is also a bijection, which is impossible because $f(\gamma) = j(f)(\gamma)$ for all $\gamma < \beta$. Similar arguments show that α is regular and strong limit. □

22. The ultrapower construction

Given an ultrafilter $\mathcal{U}$ on some cardinal κ we can form the ultrapower of V by $\mathcal{U}$, denoted by $Ult(V, \mathcal{U})$, as follows.

Let V^κ be the proper class of all κ-sequences of sets. We define an equivalence relation $\equiv_\mathcal{U}$ on V^κ by:

$$f \equiv_\mathcal{U} g \text{ if and only if } \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in \mathcal{U}.$$

Since the equivalence classes $[f]$ are proper classes, we redefine

$$[f] := \{g : g \equiv_\mathcal{U} f \text{ and } \forall h(h \equiv_\mathcal{U} f \to rank(g) \leq rank(h))\}$$

which is a set.

Now define a relation $E_\mathcal{U}$ on $V^\kappa / \equiv_\mathcal{U}$ by:

$$[f]E_{\mathcal{U}}[g] \text{ if and only if } \{\alpha < \kappa : f(\alpha) \in g(\alpha)\} \in \mathcal{U}.$$

The ultrapower $Ult(V,\mathcal{U})$ is defined as $\langle V^\kappa / \equiv_{\mathcal{U}}, E_{\mathcal{U}}\rangle$.

It is not hard to check (Łoś Theorem) that

$$Ult(V,\mathcal{U}) \models \varphi([f_1], \dots, [f_n]) \text{ iff } \{\alpha < \kappa : \varphi(f_1(\alpha), \dots, f_n(\alpha))\} \in \mathcal{U}.$$

If φ is a sentence in the language of set theory, then $Ult(V,\mathcal{U}) \models \varphi$ if and only if $V \models \varphi$. Thus, V and $Ult(V,\mathcal{U})$ are *elementarily equivalent*.

For each x, let c_x be the function on κ with constant value x. Then, the map $j : V \to Ult(V,\mathcal{U})$ given by $j(x) = [c_x]$ is an elementary embedding.

PROPOSITION 22.1. *If $\mathcal{U}$ is σ-complete, then $Ult(V,\mathcal{U})$ is well-founded.*

PROOF. First notice that for every $[f] \in Ult(V,\mathcal{U})$, the collection of all $[g]$ such that $[g]E_{\mathcal{U}}[f]$ is a set, because for each such g there is $h \in [g]$ with $rank(h) \leq rank(f)$.

Now suppose, towards a contradiction, that there is an infinite descending chain $[f_{n+1}]E_{\mathcal{U}}[f_n]$. For each n, let $X_n \in \mathcal{U}$ witness $[f_{n+1}]E_{\mathcal{U}}[f_n]$. By σ-completeness, there is $\alpha \in \bigcap_{n<\omega} X_n$. But then, $f_{n+1}(\alpha) \in f_n(\alpha)$, for all n, thus giving an infinite descending $\in$-chain, which is impossible. □

23. Measurable cardinals and elementary embeddings

THEOREM 23.1 (Keisler and Scott, 1961). *κ is measurable if and only if there exists an elementary embedding $j : V \to M$, with M transitive, such that $\kappa = crit(j)$.*

PROOF. Suppose first that κ is measurable, and let $\mathcal{U}$ be a κ-complete non-principal ultrafilter over κ. Let $j_{\mathcal{U}} : V \to Ult(V,\mathcal{U})$ be the corresponding elementary embedding. The ultrapower $Ult(V,\mathcal{U})$ is well-founded, so there is a Mostowski collapse class isomorphism $\pi : Ult(V,\mathcal{U}) \to M$, with M transitive. Then the embedding $j := \pi \circ j_{\mathcal{U}} : V \to M$ is elementary, so we only need to check that $\kappa = crit(j)$.

Let $\gamma < \kappa$ and assume $j(\beta) = \beta$ for all $\beta < \gamma$. If $\gamma < j(\gamma)$, then $[f]E_{\mathcal{U}}[c_\gamma]$, for some f such that $\pi([f]) = \gamma$. So the set $\{\alpha < \kappa : f(\alpha) \in \gamma\}$ is in $\mathcal{U}$, hence since $\mathcal{U}$ is κ-complete, f has constant value some $\beta < \gamma$ on a set in $\mathcal{U}$. But then $[f] = [c_\beta]$, and so $\gamma = \pi([f]) = \pi([c_\beta]) = j(\beta) = \beta$, which is impossible. This shows j is constant below κ. Now let id be the identity function on κ. Clearly, $[c_\beta]E_{\mathcal{U}}[id]E_{\mathcal{U}}[c_\kappa]$, for all $\beta < \kappa$. Thus, $\beta = j(\beta) < \pi([id]) < j(\kappa)$, for all $\beta < \kappa$. Hence, $\kappa < j(\kappa)$.

For the converse, suppose $j : V \to M$ is an elementary embedding, with M transitive, and with $\kappa = crit(j)$. Define $\mathcal{U}$ as follows:

$$X \in \mathcal{U} \text{ iff } X \subseteq \kappa \text{ and } \kappa \in j(X).$$

It is easy to see that $\mathcal{U}$ is an ultrafilter over κ. Notice that for every $\alpha < \kappa$, $j(\{\alpha\}) = \{\alpha\}$, and so $\mathcal{U}$ is non-principal. Let us check it is κ-complete. So let $\{X_\alpha : \alpha < \beta\} \subseteq \mathcal{U}$, some $\beta < \kappa$, and let $X := \bigcap_{\alpha<\beta} X_\alpha$. Then,

$$\kappa \in \bigcap_{\alpha<\beta} j(X_\alpha) = \bigcap_{\alpha<j(\beta)} j(X_\alpha) = j(\bigcap_{\alpha<\beta} X_\alpha) = j(X)$$

and so $X \in \mathcal{U}$. □

Let us observe that the ultrafilter $\mathcal{U}$ defined at the end of the last proof is normal. For suppose $\{X_\alpha : \alpha < \kappa\} \subseteq \mathcal{U}$. Recall that $\Delta_{\alpha<\kappa} X_\alpha$ is defined as the set $\{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} X_\beta\}$. So,

$$\kappa \in \{\alpha < j(\kappa) : \alpha \in \bigcap_{\beta<\alpha} j(X_\beta)\} = j(\Delta_{\alpha<\kappa} X_\alpha)$$

and so $\Delta_{\alpha<\kappa} X_\alpha \in \mathcal{U}$.

Suppose $\mathcal{U}$ is a κ-complete non-principal ultrafilter on κ, and let $j : V \to M \cong Ult(V, \mathcal{U})$ be the corresponding ultrapower embedding. Then

(1) $M^\kappa \subseteq M$.
(2) $\mathcal{U} \notin M$
(3) $2^\kappa < j(\kappa) < (2^\kappa)^+$

Note that (1) implies that $V_{\kappa+1} \subseteq M$, and (2) implies that $M \neq V$.

THEOREM 23.2. *If κ is measurable, then κ is weakly compact.*

PROOF. Fix a partition $f : [\kappa]^2 \to 2$. Let $\mathcal{U}$ be a κ-complete, non-principal, normal ultrafilter on κ. For each $\alpha < \kappa$, let $f_\alpha : [\kappa]^1 \to 2$ be given by: $f_\alpha(\beta) = f(\{\alpha, \beta\})$. Since $\mathcal{U}$ is an ultrafilter, for each $\alpha < \kappa$ there is $X_\alpha \in \mathcal{U}$ that is f_α-homogeneous, with constant value i_α. Let $X := \Delta\{X_\alpha : \alpha < \kappa\}$. Since $\mathcal{U}$ is normal, $X \in \mathcal{U}$. If $\alpha, \beta \in X$ and $\alpha < \beta < \kappa$, then $\beta \in X_\alpha$, and so $f(\{\alpha, \beta\}) = i_\alpha$. Let $i \in \{0, 1\}$ and $H \subseteq X$, $H \in \mathcal{U}$, be such that $i_\alpha = i$ for all $\alpha \in H$. Then $f(\{\alpha, \beta\}) = i$ for all $\alpha, \beta \in H$. □

If $\mathcal{U}$ is an ultrafilter on a regular uncountable cardinal κ, then $\mathcal{U}$ is normal if and only in for every regressive function f on a set $S \in \mathcal{U}$ there exists $S' \in \mathcal{U}$ contained in S on which f is constant.

Also recall that if $\mathcal{U}$ is a κ-complete and normal non-principal ultrafilter over κ, then it contains all club subsets of κ, and therefore every element of $\mathcal{U}$ is stationary.

Now suppose $\mathcal{U}$ is a normal κ-complete non-principal ultrafilter over κ. In $Ult(V, \mathcal{U})$, suppose $[f] E_{\mathcal{U}} [id]$. Then f is regressive on a set in $\mathcal{U}$. Hence, it is constant on a set in $\mathcal{U}$, and so $[f] = [c_\alpha]$, for some $\alpha < \kappa$.

Also, clearly $[c_\alpha] E_{\mathcal{U}} [id]$, for all $\alpha < \kappa$. Thus, we must have $\kappa = \pi([id])$.

So suppose κ is measurable and $\mathcal{U}$ is a κ-complete non-principal ultrafilter on κ which is normal. Let $j : V \to M$ be the corresponding ultrapower embedding.

Since $V_{\kappa+1} \subseteq M$, and since κ is weakly compact in V, we have that κ is also weakly compact in M. But since $\mathcal{U}$ is normal, $\kappa = \pi([id])$. Hence, in $Ult(V,\mathcal{U})$, $[id]$ is weakly compact. It follows that the set of weakly compact cardinals smaller than κ belongs to $\mathcal{U}$, and so it is stationary.

24. Strong cardinals

Recall that a measurable cardinal exists iff there exists a Σ_1-elementary (hence fully elementary) embedding $j : V \to M$, some M transitive, which is non-trivial, i.e., not the identity. The measurable cardinal is the critical point of the embedding, which we denote by $c.p.(j)$.

A word of caution: Notice that the sentence *There exists an embedding from V into M* is refutable in ZF, since V is a proper class. So, what is going on is the following: given κ measurable, i.e., given a κ-complete non-principal ultrafilter $\mathcal{U}$ over κ, we can define from $\mathcal{U}$ an elementary embedding from V into a transitive class M. And conversely, from a definable (with parameters) such an embedding with critical point κ, we can define a normal κ-complete ultrafilter on κ, which is, of course, a set. Thus, when we say that *There is measurable cardinal* is equivalent to *There exists a non-trivial elementary embedding $j : V \to M$, M transitive*, we are not asserting that this equivalence is *provable* in ZFC, as when we say that *There is an inaccessible cardinal* is equivalent to *There exists a regular cardinal κ such that $V_\kappa \prec_1 V$.*

For conciseness, whenever we write $j : V \to M$ we will always assume that j is a definable (with parameters) Σ_1-elementary embedding and M is a definable (with parameters) transitive class.

If $j : V \to M$ and $c.p.(j) = \kappa$, then the set

$$\mathcal{U} = \{X \subseteq \kappa : \kappa \in j(X)\}$$

is a normal κ-complete ultrafilter on κ. If $M_{\mathcal{U}}$ is the associated (transitive collapse of the) ultrapower and $j_{\mathcal{U}} : V \to M_{\mathcal{U}}$ is the corresponding elementary embedding, then there is an elementary embedding $k : M_{\mathcal{U}} \to M$ such that $j = k \circ j_{\mathcal{U}}$. Namely, $k([f]) = (j(f))(\kappa)$. Thus, if there is an $j : V \to M$ at all, with critical point κ, then there is one that comes from a normal κ-complete ultrafilter on κ.

We have seen that if $j : V \to M$ is the embedding that comes from a κ-complete ultrafilter on κ, then $V_{\kappa+1} \subseteq M$. On the other hand, Kunen's theorem shows that there cannot be any $j : V \to V$, other than the identity. Thus, the closer M is to V, the stronger, i.e., closer to inconsistency, is the hypothesis of the existence of a non-trivial $j : V \to M$. This opens the door to large cardinal hypotheses stronger than measurability.

DEFINITION 24.1. *(Gaifman, 1974) Let γ be an ordinal. A cardinal κ is* γ-strong *if there exists $j : V \to M$ with $c.p(j) = \kappa$ and $V_{\kappa+\gamma} \subseteq M$.*

Thus, κ is measurable iff it is 0-strong iff it is 1-strong. Since every ultrafilter on κ belongs to $V_{\kappa+2}$, if κ is 2-strong, then κ is measurable in M and, therefore, there is a measure-1 set of measurable cardinals below κ.

DEFINITION 24.2. *κ is* strong *iff it is γ-strong for every γ.*

For κ a γ-strong cardinal, witnessed by $j : V \to M$, there are two possibilities:
(1) $\gamma < j(\kappa)$
(2) $j(\kappa) \leq \gamma$

In case (2) we have $V_{j(\kappa)} \subseteq M$, and we say that κ is *superstrong*, a large-cardinal notion which has a much higher consistency strength (see Appendix 1) than a strong cardinal. So, whenever we talk about γ-strong cardinals, we will always assume that (1) is the case.

We have also seen that if $j : V \to M$ comes from a κ-complete ultrafilter $\mathcal{U}$ on κ, then $\mathcal{U} \notin M$. So, suppose that κ is 2-strong, and $j : V \to M$ is the embedding that witnesses it. Then j cannot come from a κ-complete ultrafilter on κ, since any such ultrafilter belongs to $V_{\kappa+2} \subseteq M$.

Thus, if κ is strong, then there are ever stronger embeddings with critical point κ, but for $\gamma \geq 2$, they cannot come from κ-complete ultrafilters on κ.

EXERCISE 24.3. *Show that if κ is strong, then $V \neq L[A]$ for every set A. (Hint: use an argument as in Scott's proof that the existence of a measurable cardinal implies $V \neq L$.)*

Observe that the definitions of γ-strong and strong cardinals have been given in terms of the existence of elementary embeddings of V into a transitive class. But since for $\gamma \geq 2$ these embeddings cannot come from ultrafilters on κ, for these definitions to make sense in ZFC we need to find equivalent formulations in terms of the existence of some *sets*, so that the corresponding elementary embeddings are definable from those sets. This is possible, but we shall not do it here.

25. Strongly compact cardinals

An uncountable cardinal κ is called *strongly compact* if for every set I, every κ-complete filter on I can be extended to a κ-complete ultrafilter on I.

Thus, since for κ regular the filter consisting on all subsets of κ whose complement has cardinality less than κ is κ-complete and non-principal, every strongly compact cardinal is measurable.

DEFINITION 25.1. *If $\delta \leq \kappa$ are uncountable cardinals, we say that κ is δ-*strongly compact *if for every set I, every κ-complete filter on I can be extended to a δ-complete ultrafilter on I. Thus, κ is strongly-compact iff it is κ-strongly compact.*

Notice that if κ is δ-strongly compact and λ is a cardinal greater than κ, then λ is also δ-strongly compact. Also note that if κ is regular and ω_1-strongly compact, then there exists a measurable cardinal less or equal than κ.

Suppose κ is δ-strongly compact. Let I be any non-empty set, and for every $a \in I$, let $X_a = \{x \in \mathcal{P}_\kappa(I) : a \in x\}$, where $\mathcal{P}_\kappa(I) = \{x \subseteq I : |x| < \kappa\}$. If κ is regular, then the set $\{X_a : a \in I\}$ generates a κ-complete filter on $\mathcal{P}_\kappa(I)$, which can be extended to a δ-complete ultrafilter on $\mathcal{P}_\kappa(I)$. Such an ultrafilter $\mathcal{U}$ is called a *δ-complete fine measure* on $\mathcal{P}_\kappa(I)$. The *fineness* condition is that $X_a \in \mathcal{U}$ for all $a \in I$.

We have the following characterizations of δ-strong compactness.

PROPOSITION 25.2. *The following are equivalent for any uncountable cardinals $\delta \leq \kappa$:*

(1) *κ is δ-strongly compact.*
(2) *For every α greater or equal than κ there exists an elementary embedding $j : V \to M$, with M transitive, and critical point greater or equal than δ, such that j is definable in V, and there exists $D \in M$ such that $j''\alpha := \{j(\beta) : \beta < \alpha\} \subseteq D$ and $M \models |D| < j(\kappa)$.*
(3) *For every set I there exists a δ-complete fine measure on $\mathcal{P}_\kappa(I)$.*

PROOF. (1)⇒(2): Assume κ is δ-strongly compact, and fix $\alpha \geq \kappa$. Suppose $\mathcal{U}$ is a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$. If $j_\mathcal{U} : V \to Ult(V,\mathcal{U})$ is the corresponding ultrapower embedding, then since $\mathcal{U}$ is δ-complete $Ult(V,\mathcal{U})$ is well-founded, hence isomorphic to a transitive M. Moreover, by δ-completeness, the critical point of $j_\mathcal{U}$ is greater than or equal to δ. Let $\pi : Ult(V,\mathcal{U}) \to M$ be the transitive collapsing map, and let $j = \pi \circ j_\mathcal{U}$. We claim that j satisfies the conditions of (2). For let $D := \pi([Id]_\mathcal{U})$, where $Id : \mathcal{P}_\kappa(\alpha) \to V$ is the identity map. Thus $D \in M$ and, by fineness, $j''\alpha \subseteq D$. Clearly, $Ult(V,\mathcal{U}) \models |[Id]_\mathcal{U}| < j_\mathcal{U}(\kappa)$, hence $M \models |D| < j(\kappa)$.

Thus, to prove (2) it will be enough to find, for every $\alpha \geq \kappa$, a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$. Notice that if $\kappa \leq \beta < \alpha$ and $\mathcal{U}$ is a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$, then the projection

$$\{X \subseteq \mathcal{P}_\kappa(\beta) : \{Y \in \mathcal{P}_\kappa(\alpha) : Y \cap \beta \in X\} \in \mathcal{U}\}$$

is a δ-complete fine measure on $\mathcal{P}_\kappa(\beta)$. So fix $\alpha \geq \kappa$ and assume, without loss of generality, that α is regular.

If κ is regular, then we have already observed above that a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$ does exist. So suppose κ is singular. Then κ^+ is regular and also δ-strongly compact. So let $\mathcal{U}^*$ be a δ-complete fine measure on $\mathcal{P}_{\kappa^+}(\alpha)$, and let $j_{\mathcal{U}^*} : V \to Ult(V,\mathcal{U}^*)$ be the ultrapower embedding, $\pi : Ult(V,\mathcal{U}^*) \cong M$ the transitive collapse, and $j := \pi \circ j_{\mathcal{U}^*}$. Note that the critical point of j is greater than or equal to δ. Letting $D := \pi([Id]_{\mathcal{U}^*})$, where $Id : \mathcal{P}_{\kappa^+}(\alpha) \to V$ is the identity map, we have that $D \in M$, $j"\alpha \subseteq D$, and $M \models$ "$|D| < j(\kappa^+) = j(\kappa)^+$".

Let $\beta = sup(j"\alpha)$. So, $\beta \cap D$ is cofinal in β. Hence, in M, the cofinality of β is at most $j(\kappa)$. And in fact, since $M \models$ "$j(\kappa)$ is singular", $cof(\beta) < j(\kappa)$.

In M, let C be a closed unbounded subset of $cof(\beta)$. Observe that $j"\alpha$ is an ω-closed subset of β. So, since $cof(\beta)$ is uncountable, $C \cap j"\alpha$ is unbounded in β. Hence, $I := \{\gamma < \alpha : j(\gamma) \in C\}$ is unbounded in α, and so $|I| = \alpha$.

Now define an ultrafilter $\mathcal{U}$ on $\mathcal{P}_\kappa(I)$ as follows:

$$X \in \mathcal{U} \text{ if and only if } X \subseteq \mathcal{P}_\kappa(I) \text{ and } j^*(I) \cap C \in j^*(X).$$

One can readily check that $\mathcal{U}$ is a δ-complete fine measure on $\mathcal{P}_\kappa(I)$ which, since $|I| = \alpha$, naturally induces a δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$.

(2)⇒(3): Without loss of generality, we may assume I is some ordinal α greater than or equal to κ. Given $j : V \to M$ and D as in (2), for α, define $\mathcal{U}$ in V by:

$$X \in \mathcal{U} \text{ if and only if } X \subseteq \mathcal{P}_\kappa(\alpha) \text{ and } D \in j(X).$$

Since $M \models |D| < j(\kappa)$, $\mathcal{U}$ is well-defined. It is easy to check that $\mathcal{U}$ is an δ-complete fine measure on $\mathcal{P}_\kappa(\alpha)$.

(3)⇒(1): Suppose F is a κ-complete filter over some set I. We may assume that F is actually a filter over $\alpha = |I|$. Let $\mathcal{U}$ be a δ-complete fine measure on $\mathcal{P}_\kappa(F)$, and let $j : V \to M \cong Ult(V,\mathcal{U})$ be the corresponding ultrapower embedding, with M transitive. Let $\pi : Ult(V,\mathcal{U}) \to M$ be the transitive collapsing map, and set $D = \pi([Id]_\mathcal{U})$. By fineness, $j''F \subseteq D$. And clearly $M \models |D| < j(\kappa)$.

In M, $j(F)$ is $j(\kappa)$-complete. So there exists $a \in \bigcap(j(F) \cap D)$. Let $\mathcal{V}$ be given by:

$$X \in \mathcal{V} \text{ if and only if } X \subseteq \alpha \text{ and } a \in j(X).$$

It is easy to see that $\mathcal{V}$ is a δ-complete ultrafilter on α. And it contains F, for if $X \in F$, then $j(X) \in D \cap j(F)$, and therefore $a \in j(X)$. □

If λ is the least measurable cardinal and κ is ω_1-strongly compact, κ not necessarily regular, then κ is λ-strongly compact. For if $\mathcal{U}$ is a ω_1-complete ultrafilter on a set I that is not λ-complete, then there is a partition $\{X_\alpha : \alpha < \beta\}$ of I, some $\beta < \lambda$, such that none of the X_α belongs to $\mathcal{U}$. But then the set $\{X \subseteq \beta : \bigcup\{X_\alpha : \alpha \in X\} \in \mathcal{U}\}$ is a non-principal ω_1-complete ultrafilter on β, contradicting the minimality of λ.

Thus if κ is ω_1-strongly compact and is also the first measurable, a consistent situation as shown by Magidor, then κ is in fact strongly compact.

Lecture IV

26. Supercompact cardinals

In the spirit of extending naturally the notion of measurable cardinal, as given by an elementary embedding form V into some transitive class M, by requiring that M is close to V, one has the following notion of large cardinal.

Definition 26.1 (Solovay, Reinhardt). *Let γ be an ordinal. A cardinal κ is* γ-supercompact *if there exists $j : V \to M$ with c.p.$(j) = \kappa$ and $M^\gamma \subseteq M$.*

It can be shown (see [**2**], p. 323) that if κ is γ-supercompact, say witnessed by $j : V \to M$, then for some $n < \omega$, the nth-iterate of j, call it j^n, also witnesses the γ-supercompactness of κ and, moreover, $\gamma < j^n(\kappa)$. Thus, we may, and will, require in the defintion of γ-supercompactness that $\gamma < j(\kappa)$.

Thus, κ is measurable iff it is γ-supercompact for some (for all) $\gamma < \kappa^+$.

Suppose that κ is 2^κ-supercompact, witnessed by $j : V \to M$. Let $\mathcal{U}$ be the ultrafilter derived from j, i.e., $X \in \mathcal{U}$ iff $X \subseteq \kappa$ and $\kappa \in j(X)$. Since $M^{2^\kappa} \subseteq M$, $\mathcal{U} \in M$. Hence, κ is measurable in M and, therefore, the set of measurable cardinals below κ belongs to $\mathcal{U}$.

Definition 26.2. *κ is* supercompact *if it is γ-supercompact for all γ.*

We will later see that supercompactness is a very strong large cardinal notion, in particular, if κ is supercompact, then there are many $\lambda < \kappa$ such that in V_λ there is a proper class of measurable cardinals.

If $j : V \to M$ witnesses that κ is κ^+-supercompact, then since $j''\kappa^+ \notin M$, j cannot come from a κ-complete ultrafilter on κ. And conversely, if $j : V \to M$ comes from a κ-complete ultrafilter on κ, then j does not witness the κ^+-supercompactness of κ.

Since we have defined the notion of γ-supercompactness only in terms of elementary embeddings of the universe into a transitive class, we want now to find, as in the case of measurable cardinals, an equivalent formulation in terms of the existence of some sets so that the corresponding elementary embeddings will be definable from those sets. In the case of a measurable cardinal κ, the sets were ultrafilters on κ. Now we know the embeddings cannot come (for $\gamma \geq \kappa^+$) from ultrafilters on κ, but perhaps they may come from ultrafilters on some other set.

PROPOSITION 26.3. *Suppose $\mathcal{U}$ is a σ-complete ultrafilter over a set A and $j : V \to M$ is the corresponding elementary embedding. Then for every ordinal γ, $j''\gamma \in M$ iff $M^\gamma \subseteq M$.*

Recall that if κ is measurable and $j : V \to M$ has critical point κ, then $j''\kappa = \kappa$ and $M^\kappa \subseteq M$. Then we defined the ultrafilter associated to j as the collection of all $X \subseteq \kappa$ such that $\kappa \in j(X)$. So, now suppose $j : V \to M$ witnesses the γ-supercompactness of κ. Since $M^\gamma \subseteq M$, it seems only natural to define an ultrafilter associated to j, call it $\mathcal{U}$, as:

$$X \in \mathcal{U} \text{ iff } X \subseteq [\gamma]^\kappa \text{ and } j''\gamma \in j(X).$$

The following can be easily checked:

PROPOSITION 26.4.

(1) *$\mathcal{U}$ is a κ-complete ultrafilter on $[\gamma]^{<\kappa} := \{X \subseteq \gamma : |X| < \kappa\}$.*
(2) *$\mathcal{U}$ is* fine*, i.e., for every $\alpha < \gamma$, $\{X \in [\gamma]^{<\kappa} : \alpha \in X\} \in \mathcal{U}$.*
(3) *$\mathcal{U}$ is* normal*, i.e., if $\langle X_\alpha : \alpha < \gamma\rangle$ is a sequence of sets from $\mathcal{U}$, then its diagonal intersection $\Delta_{\alpha<\gamma} X_\alpha := \{x : x \in \bigcap_{\alpha \in x} X_\alpha\}$ belongs to $\mathcal{U}$.*

PROOF. (1): First notice that since $\gamma < j(\kappa)$, $j''\gamma \in j([\gamma]^{<\kappa}) = ([j(\gamma)]^{<j(\kappa)})^M$, and so $[\gamma]^{<\kappa} \in \mathcal{U}$. The rest is straightforward.

(2): Need to check that $j''\gamma \in j(\{X \in [\gamma]^{<\kappa} : \alpha \in X\}) = \{X \in [j(\gamma)]^{<j(\kappa)} : j(\alpha) \in X\}$. But since $\gamma < j(\kappa)$, this is obvious.

(3): Need to check that $j''\gamma \in j(\{x \in [\gamma]^{<\kappa} : x \in \bigcap_{\alpha\in x} X_\alpha\}) = \{x \in [j(\gamma)]^{<j(\kappa)} : x \in \bigcap_{\alpha \in x} j(X_\alpha)\}$. Since $\gamma < j(\kappa)$, this is obvious. □

EXERCISE 26.5. *Show that if $\mathcal{U}$ is as above and $j : V \to M$ is the associated elementary embedding, then $[id] = j''\gamma$. Hence, for every function f on $[\gamma]^{<\kappa}$, $[f] = (j(f))(j''\gamma)$.*

EXERCISE 26.6. *Show that if $\mathcal{U}$ is a fine measure on $[\gamma]^{<\kappa}$, then $\mathcal{U}$ is normal iff whenever $f : [\gamma]^{<\kappa} \to V$ is such that $f(X) \in X$ for almost all X, then f is constant for almost all X. (Hint: Use the same argument as for measures on a cardinal κ. Fineness plays the role in this case as the fact that, in the case of measures on κ, final segments have measure 1.)*

DEFINITION 26.7. *A* supercompact measure *on $[\gamma]^\kappa$ is a κ-complete, fine and normal ultrafilter on $[\gamma]^{<\kappa}$.*

THEOREM 26.8. *If $\kappa \leq \gamma$, then κ is γ-supercompact iff there is a supercompact measure on $[\gamma]^\kappa$.*

PROOF. We have just proved one direction, namely, if $j : V \to M$ witnesses the γ-supercompactness of κ, then $\mathcal{U} = \{X \subseteq [\gamma]^{<\kappa} : j''\gamma \in j(X)\}$ is a supercompact measure.

Conversely, if $\mathcal{U}$ is a supercompact measure on $[\gamma]^\kappa$, let $j = j_{\mathcal{U}} : V \to M$ be the associated elementary embedding. Let us first check that $j(\kappa) > \gamma$. Let f be the function that assigns to every element of $[\gamma]^{<\kappa}$ its order type, i.e., $f(X) = o.t.(X)$. We have (see Exercise 26.5) that $[f] = (j(f))(j''\gamma) = o.t.(j''\gamma) = \gamma$. Hence, since $o.t.(X) < \kappa$ for all $X \in [\gamma]^{<\kappa}$, we have $\gamma < j(\kappa)$.

To see that $M^\gamma \subseteq M$ it is enough to show, by Proposition 26.3, that $j''\gamma \in M$. For each $\alpha < \gamma$, let $f_\alpha : [\gamma]^{<\kappa} \to OR$ be such that $j(\alpha) = [f_\alpha]$. Let now f be the function with domain $[\gamma]^{<\kappa}$ given by: $f(X) = \{f_\alpha(X) : \alpha \in X\}$. We claim that $[f] = j''\gamma$. By fineness of $\mathcal{U}$, for every $\alpha < \gamma$, $\alpha \in X$ for almost all $X \in [\gamma]^{<\kappa}$. Hence, for almost all X, $f_\alpha(X) \in f(X)$, and so $[f_\alpha] \in [f]$. On the other hand, if $[g] \in [f]$, then $g(X) \in f(X)$ for almost all X, and so for almost all X, $g(X) = f_\alpha(X)$ for some $\alpha \in X$. By normality applied to the function $g'(X) =$ the α such that $g(X) = f_\alpha(X)$ (see Exercise 26.6), there is $\alpha < \gamma$ such that $g(X) = f_\alpha(X)$ for almost all X, and so $[g] = [f_\alpha] = j(\alpha)$. □

Exercise 26.9. *If $\mathcal{U}$ is a supercompact measure on $[\gamma]^\kappa$, then $\mathcal{U}$ contains every closed and unbounded subset of $[\gamma]^{<\kappa}$.*

If κ is supercompact, then $V_\kappa \prec_2 V$.

The last theorem shows that if κ is supercompact, then it is strongly compact. However, the converse is not true.

Theorem 26.10 (Magidor, 1976).

(1) *If κ is supercompact, then there is a forcing extension of V in which κ is supercompact and is also the least strongly compact cardinal.*
(2) *If κ is strongly compact, then there is a forcing extension of V in which it is still strongly compact and is also the first measurable cardinal.*

27. Extendible cardinals

A cardinal κ is λ–*extendible* if there is an elementary embedding $j : V_\lambda \to V_\mu$, some μ, with critical point κ and such that $j(\kappa) > \lambda$. And κ is *extendible* if it is λ–extendible for all $\lambda > \kappa$.

The next lemma implies that every extendible cardinal is supercompact.

Lemma 27.1 (M. Magidor). *Suppose $j : V_\lambda \to V_\mu$ is elementary, λ is a limit ordinal, and κ is the critical point of j. Then κ is $< \lambda$-supercompact.*

Proof. Fix $\gamma < \lambda$ and define

$$\mathcal{U}_\gamma = \{X \subseteq \mathcal{P}_\kappa(\gamma) : j''\gamma \in j(X)\}.$$

Note that this makes sense if $j(\kappa) > \gamma$, in which case it is easy to check that $\mathcal{U}_\gamma$ is a κ-complete, fine, and normal measure. Otherwise, let $j^1 = j$ and $j^{m+1} = j \circ j^m$. If $j^m(\kappa) > \gamma$ for some m, then define $\mathcal{U}_\gamma$ using j^m instead of j. But such an m does exist, for otherwise $\delta := sup_m(j^m(\kappa)) \leq \gamma < \lambda$, and then since $j(\delta) = \delta$ we would have $j \restriction V_{\delta+2} : V_{\delta+2} \to V_{\delta+2}$ is elementary with critical point κ, contradicting Kunen's Theorem. □

If κ is extendible, then the set of supercompact cardinals smaller than κ is stationary.

28. Vopenka's Principle

Vopěnka's Principle (VP) (after Petr Vopěnka, circa 1960) states that for every proper class $\mathcal{C}$ of structures of the same type, there exist $A \neq B$ in $\mathcal{C}$ such that A is elementarily embeddable into B.

VP can be formulated in the first-order language of set theory as an axiom schema, i.e., as an infinite set of axioms, one for each formula with two free variables. Formally, for each such formula $\varphi(x, y)$ one has the axiom:

$$\forall x[(\forall y \forall z(\varphi(x,y) \wedge \varphi(x,z) \rightarrow y \text{ and } z \text{ are structures of the same type}) \wedge$$
$$\forall \alpha \in OR\, \exists y(rank(y) > \alpha \wedge \varphi(x,y)) \rightarrow$$
$$\exists y \exists z(\varphi(x,y) \wedge \varphi(x,z) \wedge y \neq z \wedge \exists e(e : y \rightarrow z \text{ is elementary}))].$$

Henceforth, VP will be understood as this axiom schema.

The theory ZFC plus VP implies, for instance, that the class of extendible cardinals is stationary, i.e., every definable club proper class contains an extendible cardinal. And its consistency is known to follow from the consistency of ZFC plus the existence of an almost-huge cardinal (see [**2**], or [**1**]).

28.1. The H_κ. Every set is contained in a smallest transitive set, called its *transitive closure*. The transitive closure of a set A, denoted by $TC(A)$ consists of all elements of A, the elements of elements of A, the elements of elements of elements of A, and so on.

For an infinite cardinal κ, H_κ is the set of all sets having transitive closure of cardinality $< \kappa$. Thus, $H_\omega = V_\omega$. We always have $H_\kappa \subseteq V_\kappa$. But $H_{\omega_1} \neq V_{\omega_1}$, as e.g., $\mathcal{P}(\omega) \in V_{\omega+2} \setminus H_{\omega_1}$. Note that all H_κ are transitive.

Similarly as with the V_α, the H_κ also form a cumulative hierarchy: if $\kappa \leq \lambda$, then $H_\kappa \subseteq H_\lambda$, and if κ is a limit cardinal, then $H_\kappa = \bigcup_{\lambda<\kappa} H_\lambda$. Finally, $V = \bigcup_{\kappa \in CARD} H_\kappa$.

There is a closed proper class of cardinals C such that $V_\kappa = H_\kappa$, for every $\kappa \in C$.

If κ is inaccessible, then $V_\kappa = H_\kappa$.

28.2. Variants of VP. Let us consider the following variants of VP, the first one apparently much stronger than the second.

We say that a class $\mathcal{C}$ is $\mathbf{\Sigma_n}$ ($\mathbf{\Pi_n}$) if it is definable, with parameters, by a Σ_n (Π_n) formula of the language of set theory. If no parameters are involved, then we use the lightface types Σ_n (Π_n).

Definition 28.1. *If Γ is one of $\mathbf{\Sigma_n}$, $\mathbf{\Pi_n}$, some $n \in \omega$, and κ is an infinite cardinal, then we write $VP(\kappa, \Gamma)$ for the following assertion:*

For every Γ proper class $\mathcal{C}$ of structures of the same type τ such that both τ and the parameters of some Γ-definition of $\mathcal{C}$, if any, belong to H_κ, $\mathcal{C}$ *reflects below*

κ, i.e., for every $B \in \mathcal{C}$, there exists $A \in \mathcal{C} \cap H_\kappa$ that is elementarily embeddable into B.

If Γ *is one of* $\mathbf{\Sigma_n}$, $\mathbf{\Pi_n}$, *or* Σ_n, Π_n, *some* $n \in \omega$, *we write* $VP(\Gamma)$ *for the following statement:*

For every Γ proper class $\mathcal{C}$ of structures of the language of set theory with one (equivalently, finitely-many) additional 1-ary relation symbol(s), there exist distinct A and B in $\mathcal{C}$ with an elementary embedding of A into B.

VP for $\mathbf{\Sigma_1}$ classes is a consequence of ZFC. In fact, the following holds.

THEOREM 28.2. *If* κ *is an uncountable cardinal, then every (not necessarily proper) class* $\mathbb{C}$ *of structures of the same type* $\tau \in H_\kappa$ *which is* Σ_1 *definable, with parameters in* H_κ*, reflects below* κ*. Hence,* $VP(\kappa, \mathbf{\Sigma_1})$ *holds for every uncountable cardinal* κ*.*

PROOF. Fix an uncountable cardinal κ and a class $\mathcal{C}$ of structures of the same type $\tau \in H_\kappa$, definable by a Σ_1 formula with parameters in H_κ.

Given $B \in \mathcal{C}$, let λ be a regular cardinal greater than κ, with $B \in H_\lambda$, and let N be an elementary substructure of H_λ, of cardinality less than κ, which contains B and the transitive closure of $\{\tau\}$ together with the parameters involved in some Σ_1 definition of $\mathcal{C}$.

Let A and M be the transitive collapses of B and N, respectively, and let $j : M \to N$ be the collapsing isomorphism. Then $A \in H_\kappa$, and $j \restriction A : A \to B$ is an elementary embedding. Observe that $j(\tau) = \tau$. So, since Σ_1 formulas are upwards absolute for transitive models, and since $M \models A \in \mathcal{C}$, we have that $A \in \mathcal{C}$. $\square$

In contrast, Vopěnka's Principle for Π_1 proper classes implies the existence of supercompact cardinals.

THEOREM 28.3. *If* $VP(\Pi_1)$ *holds, then there exists a supercompact cardinal.*

PROOF. Let $\mathcal{C}$ be the class of structures of the form $\langle V_{\lambda+2}, \in, \alpha, \lambda \rangle$, where λ is the least limit ordinal greater than α such that no $\kappa \leq \alpha$ is $< \lambda$-supercompact.

We claim that $\mathcal{C}$ is Π_1 definable without parameters. For $X \in \mathcal{C}$ if and only if $X = \langle X_0, X_1, X_2, X_3 \rangle$, where

(1) X_2 is an ordinal
(2) X_3 is a limit ordinal greater than X_2
(3) $X_0 = V_{X_3+2}$
(4) $X_1 = \in \restriction X_0$
(5) And the following hold in $\langle X_0, X_1 \rangle$:
  (a) $\forall \kappa \leq X_2 (\kappa$ is not $< X_3$-supercompact$)$
  (b) $\forall \mu (\mu \text{ limit} \wedge X_2 < \mu < X_3 \to \exists \kappa \leq X_2 (\kappa \text{ is } < \mu\text{-supercompact}))$.

If there is no supercompact cardinal, then $\mathcal{C}$ is a proper class. So by $VP(\Pi_1)$, there exist $\langle V_{\lambda+2}, \in, \alpha, \lambda \rangle \neq \langle V_{\mu+2}, \in, \beta, \mu \rangle$ and an elementary embedding

$$j : \langle V_{\lambda+2}, \in, \alpha, \lambda \rangle \to \langle V_{\mu+2}, \in, \beta, \mu \rangle.$$

Since j must send α to β and λ to μ, j is not the identity. Hence by Kunen's theorem we must have $\lambda < \mu$, and therefore also $\alpha < \beta$. So, j has critical point some $\kappa \leq \alpha$. It now follows by Lemma 27.1 that κ is $< \lambda$-supercompact. But this is impossible because $\langle V_{\lambda+2}, \in, \alpha, \lambda \rangle \in \mathcal{C}$. □

We give next a strong converse to Theorem 28.3.

THEOREM 28.4. *Suppose that* $\mathcal{C}$ *is a* $\boldsymbol{\Sigma_2}$ *(not necessarily proper) class of structures of the same type* τ*, and suppose that there exists a supercompact cardinal* κ *larger than the rank of the parameters that appear in some* Σ_2 *definition of* $\mathcal{C}$*, and with* $\tau \in V_\kappa$*. Then for every* $B \in \mathcal{C}$ *there exists* $A \in \mathcal{C} \cap V_\kappa$ *that is elementarily embeddable into* B*.*

PROOF. Fix a Σ_2 formula $\varphi(x, y)$ and a set b such that $\mathcal{C} = \{B : \varphi(B, b)\}$, and suppose that κ is a supercompact cardinal with $b \in V_\kappa$. Fix $B \in \mathcal{C}$, and let $\lambda \in C^{(2)}$ be greater than rank(B). Let $j \colon V \to M$ be an elementary embedding with M transitive and critical point κ, such that $j(\kappa) > \lambda$ and M is closed under λ-sequences. Thus, B and $j \restriction B : B \to j(B)$ are in M, and also $V_\lambda \in M$. Hence $V_\lambda \preceq_1 M$. Moreover, since $j(\tau) = \tau$, $j(B)$ is a tructure of type τ, and $j \restriction B$ is an elementary embedding.

Since $V_\lambda \preceq_2 V$, $V_\lambda \models \varphi(B, b)$. And since Σ_2 formulas are upwards absolute between V_λ and M, $M \models \varphi(B, b)$.

Thus, in M it is true that there exists $X \in M_{j(\kappa)}$ such that $\varphi(X, b)$, namely B, and there exists an elementary embedding $e \colon X \to j(B)$, namely $j \restriction B$. Therefore, by elementarity, the same holds in V; that is, there exists $X \in V_\kappa$ such that $\varphi(X, b)$, and there exists an elementary embedding $e \colon X \to B$. □

The following corollary gives a characterization of Vopěnka's principle for Π_1 and Σ_2 classes in terms of supercompactness.

COROLLARY 28.5. *The following are equivalent:*

(1) $VP(\Pi_1)$.
(2) *There exists a supercompact cardinal.*

We shall give next a characterization of supercompactness in terms of a natural principle of reflection.

Recall from Definition 28.1 that a cardinal κ *reflects* a class of structures $\mathcal{C}$ of the same type if for every $B \in \mathcal{C}$ there exists $A \in \mathbb{C} \cap H_\kappa$ which is elementary embeddable into B.

THEOREM 28.6 (Magidor [**4**]). *If* κ *is the least cardinal that reflects the* Π_1 *proper class* $\mathcal{C}$ *of structures of the form* $\langle V_\lambda, \in \rangle$*, then* κ *is supercompact.*

PROOF. For each λ greater than κ there is $\alpha < \kappa$ and an elementary embedding

$$j_\lambda : \langle V_\alpha, \in \rangle \to \langle V_\lambda, \in \rangle.$$

Let α be be the least ordinal for which there is such an embedding for a proper class of limit λ. We may assume that the j_λ are not the identity, for if they were the identity for a proper class of λ, then V_α would be an elementary substructure of V, which is impossible because α is definable. We may also assume that the critical point of all these embeddings is the same, say β, and that β is the least such. Moreover, we may assume that the image of β is always the same, for otherwise for a proper class of λ the identity embedding $j_\lambda \restriction V_\beta$ would witness that V_β is an elementary substructure of $V_{j_\lambda(\beta)}$, with the $j_\lambda(\beta)$ forming a proper class, which in turn would imply that V_β is an elementary substructure of V, an impossibility since β is definable.

So let δ be least such that for a proper class C of limit λ the α is the same, j_λ is not the identity, the critical point β is the same, and $j_\lambda(\beta) = \delta$. By Lemma 27.1, β is $< \alpha$-supercompact. Hence, by elementarity of the j_λ, δ is $< \lambda$-supercompact for all $\lambda \in C$, and therefore δ is supercompact. Thus $\delta \geq \kappa$, because δ reflects $\mathbb{C}$, by Theorem 28.4, and κ is the least cardinal that does this. So suppose, aiming for a contradiction, that $\delta > \kappa$. By Theorem 28.4, δ reflects the proper class of structures of the form $\langle V_\lambda, \in, \gamma \rangle$, where λ is a limit ordinal and $\gamma < \lambda$, which is Π_1. So, similarly as before, there are fixed $\gamma < \alpha < \kappa$ and elementary embeddings $k_\lambda : \langle V_\alpha, \in, \gamma \rangle \to \langle V_\lambda, \in, \kappa \rangle$, for a proper class of limit λ, all with the same critical point, and whose image of the critical point is some fixed ordinal less or equal than κ, contradicting the minimality of δ. □

The last two theorems yield the following characterizations of the first supercompact cardinal.

Corollary 28.7. *The following are equivalent:*

1. *κ is the first supercompact cardinal.*
2. *κ is the least ordinal that reflects the Π_1 class of structures of the form $\langle V_\lambda, \in \rangle$, λ an ordinal.*

29. The strongest large cardinals

A cardinal κ is called a *Reinhardt cardinal* if there exists an elementary embedding $j : V \to V$ with critical point κ.

Theorem 29.1 (Kunen, 1971). *Reinhardt cardinals don't exist.*

In fact, Kunen proves that there doesn't exist any non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$.

The existence of an elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$ is one of the strongest large cardinal principles not known to be inconsistent.

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